Connections and Related Integral Structures on the Universal Extension of an Elliptic Curve

by Shinichi Mochizuki

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§0. Introduction

In this paper, we continue our study of the *Hodge-Arakelov theory of elliptic curves*, initiated in [Mzk1], [Mzk2]. The essence of this theory lies in thinking of the comparison isomorphism of the (complex or p-adic) Hodge theory of an elliptic curve as a *restriction morphism from functions on the de Rham cohomology to functions on some sort of "torsion points" inside the de Rham cohomology* (cf. [Mzk1], Introduction). This function-theoretic point of view allowed us in [Mzk1] to *discretize* the usual complex and p-adic Hodge theories of an elliptic curve into a global, Arakelov-theoretic *"Hodge-Arakelov Comparison Isomorphism"* (cf. [Mzk1]).

The first main goal of the present paper is to clear up the confusion surrounding the discussion of the *"étale integral structure"* on the universal extension of an elliptic curve, which was introduced in [Mzk1]. This integral structure may be described as follows. Over a formal neighborhood of the point at infinity on the moduli stack of elliptic curves, the tautological elliptic curve E looks like $\mathbf{G}_{\mathrm{m}}$, while its *universal extension* $E^\dagger$ (roughly speaking: the moduli space of degree zero line bundles on E equipped with a connection) may be described as the product $\mathbf{G}_{\mathrm{m}} \times \mathbf{A}^1$ of $\mathbf{G}_{\mathrm{m}}$ with the affine line. If we write "T" for the standard coordinate on this affine line, then near infinity, the *standard integral structure* on $E^\dagger$ may be described as that given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbf{G}_{\mathrm{m}}} \cdot T^r$$

while the *étale integral structure* on $E^\dagger$ is given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbf{G}_{\mathrm{m}}} \cdot \binom{T}{r}$$

(where $\binom{T}{r} \stackrel{\mathrm{def}}{=} \frac{1}{r!} T(T-1) \cdot \ldots \cdot (T-(r-1))$). Since this integral structure forms an algebra which is compatible with the group scheme structure of $E^\dagger$, it corresponds to a geometric object $E^\dagger_{\mathrm{et}}$ such that $E^\dagger_{\mathrm{et}} \otimes \mathbf{Q} = E^\dagger \otimes \mathbf{Q}$.

Although the above definition of $E^\dagger_{\mathrm{et}}$ is only valid near infinity, this integral structure may, in fact, be extended over the *entire moduli stack of elliptic curves.* This fact is discussed in [Mzk1], Chapter V, §3, but the proof given there is incomplete. Thus, in §1 of the present paper, we give a *complete proof of the extendability of this integral structure* over the entire moduli stack of elliptic curves (cf. Theorem 1.3). Next, in §2, we analyze the *p-adic structure* of $E^\dagger_{\mathrm{et}}$ in the case of a p-adic elliptic curve whose reduction modulo p is *ordinary.* The main result of this analysis is an isomorphism (cf. Theorem 2.2)

$$(E^\dagger_{\mathrm{et}})^\wedge \cong E^{\mathrm{F}^\infty}$$

between the p-adic completion of $E^{\dagger}_{\mathrm{et}}$ and a certain object $E^{\mathrm{F}^{\infty}}$ obtained by considering composites of the "Verschiebung morphism associated to an ordinary elliptic curve."

In fact, ultimately, in both the theory of [Mzk1] and of the present paper, we will need to make use not only of $E^{\dagger}$ and $E^{\dagger}_{\mathrm{et}}$, but of certain natural torsors — which we refer to as *Hodge torsors* — over $E^{\dagger}$ and $E^{\dagger}_{\mathrm{et}}$. The Hodge torsors E^{*} may be interpreted functorially over E as the *moduli spaces of connections on certain ample line bundles* on E (where $*$ denotes the ample line bundle in question). The elementary theory of Hodge torsors and their (partial) compactifications is the topic of §3. In fact, over $\mathbf{Q}$, we have natural identifications $E^{*} \otimes \mathbf{Q} = E^{\dagger} \otimes \mathbf{Q}$, i.e., Hodge torsors may be thought of as being certain integral structures on $E^{\dagger}$. In a neighborhood of infinity, (relative to the notation introduced above) this integral structure may be described as that in which the "T^r" (in the case of $E^{\dagger}$) are replaced by "$(T - (i_{\chi}/n))^r$" (where the rational number i_{χ}/n is an invariant determined by the ample line bundle in question).

Just as in the case of the universal extension $E^{\dagger}$, *the Hodge torsors also admit étale integral structures*, i.e., the integral structures in which the "$(T - (i_{\chi}/n))^r$" are replaced by

$$\binom{T - (i_{\chi}/n)}{r}$$

These integral structures are, in fact, necessary not just in the theory of the present paper, but also in the theory of [Mzk1]. In [Mzk1], however, no proof is given of the fact that the above definition near infinity (i.e., $\binom{T-(i_{\chi}/n)}{r}$) extends neatly over the entire moduli stack of elliptic curves. In the present paper, however, we give a complete proof of this result (cf. Theorem 4.3). It turns out that this extendability result for Hodge torsors is somewhat more difficult than its universal extension analogue (i.e., Theorem 1.3), *especially at the prime* $p = 2$. In particular, at the prime $p = 2$, in order to complete the proof of Theorem 4.3, *it is necessary to use the nontrivial theory of connections and higher p-curvatures developed in the second half of the present paper.*

In §5, we prove that the pair $(E^{*}_{\mathrm{et}}, \mathcal{L}_{E^{*}_{\mathrm{et}}})$ (where $*$ may be set equal either to $\dagger$ — in which case $E^{*}_{\mathrm{et}} \stackrel{\mathrm{def}}{=} E^{\dagger}_{\mathrm{et}}$ — or to an ample line bundle (satisfying certain conditions) — in which case E^{*}_{et} is to denote the corresponding Hodge torsor equipped with its étale integral structure) consisting of E^{*}_{et} together with the pull-back to E^{*}_{et} of an ample line bundle $\mathcal{L}$ on the original elliptic curve E admits a *connection* satisfying certain natural functorial properties (cf. Theorems 5.2, 5.3; Corollary 8.3). That is to say, put another way, this means that the pair $(E^{*}_{\mathrm{et}}, \mathcal{L}_{E^{*}_{\mathrm{et}}})$ forms a *crystal valued in the category of "polarized schemes"* (i.e., schemes equipped with an ample line bundle) over the base. This generalizes the classical result that the universal extension itself $E^{\dagger}$ forms a crystal in schemes over the base. From another point of view, this result may be regarded as the scheme-theoretic analogue of the complex analytic fact that if E is a Riemann surface equipped with an ample line bundle $\mathcal{L}$, then (if we denote the underlying real analytic objects corresponding to various complex analytic objects by means of a subscript "$\mathbf{R}$," then) not only $E_{\mathbf{R}}$, but the pair $(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ is, in fact, a *topological invariant* (i.e., does not vary as the complex moduli of E vary) of E.

In fact, it is useful here to recall that in some sense:

> *The essential spirit of the "Hodge-Arakelov Theory of Elliptic Curves" (studied in [Mzk1], [Mzk2], and the present paper) may be summarized as being the Hodge theory of the pairs $(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ (at archimedean primes), $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ (at non-archimedean primes), as opposed to the "usual Hodge theory of an elliptic curve" which may be thought of as the Hodge theory of $E_{\mathbf{R}}$ (at archimedean primes) or E^*_{et} (at non-archimedean primes).*

In this connection, we note that the "Hodge-Arakelov theory of an elliptic curve at an archimedean prime" is discussed/reviewed in detail in [Mzk1], Chapter VII, §4. On the other hand, the very direct and explicit relationship between $E^{\dagger}_{\mathrm{et}}$ at archimedean primes and the "classical" p-adic Hodge theory of an elliptic curve may be seen in the theory of §2 of the present paper.

Once the canonical connection on the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ is constructed in §5, we then proceed to prove in §6, 7, what may be regarded as *Local Hodge-Arakelov Comparison Isomorphisms*, i.e., local versions (that is to say, versions relating to a formal neighborhood of a (scheme-valued) point of the moduli stack of elliptic curves) of the "discrete Hodge-Arakelov Comparison Isomorphism" of [Mzk1]. The first such local comparison isomorphism is given in §6 (cf. Theorem 6.2), and is referred to as the *"Schottky-Theoretic Hodge-Arakelov Comparison Isomorphism,"* since it involves the Schottky uniformization of a degenerating elliptic curve. The main technical result underlying Theorem 6.2 is the *explicit computation* (cf. Theorem 6.1) of the connection constructed in §5 near infinity (i.e., for degenerating elliptic curves). The second local comparison isomorphism is given in §7 (cf. Theorem 7.6; Corollary 8.3) and concerns formal neighborhoods of *nondegenerating elliptic curves*.

Another interesting way to think of these local comparison isomorphisms is as results which allow one to give natural *theta expansions* of sections of an ample line bundle on an elliptic curve. Near infinity (cf. §6), this theta expansion is simply the classical one. At smooth points (cf. §7), however, this expansion appears to be *new*, and gives rise to a sort of *crystalline theta expansion* of sections of an ample line bundle on an elliptic curve (cf. the discussion of §7 for more details).

In some sense, the comparison isomorphisms of Theorem 6.2 and Corollary 7.6 may be regarded as *prototypes* of the comparison isomorphism of [Mzk1]. Put another way, it seems natural to regard the comparison isomorphism of [Mzk1] as a sort of *"extension via discretization"* of the *local* comparison isomorphisms of §6, 7 (which, as discussed above, concern *formal neighborhoods* of points in the moduli stack) over the entire moduli stack of log elliptic curves. In fact, since the proofs of the local comparison isomorphisms of the present paper are, in many respects, *technically much simpler than the proof of the* *main result of [Mzk1], many readers may wish to study the present paper as a sort of "introduction" to the main idea behind the theory of [Mzk1].*

Yet another interesting aspect of the local comparison isomorphisms of the present paper — especially Theorem 6.2 — is that *Theorem 6.2 further justifies the assertion of the author in [Mzk1] that the "arithmetic Kodaira-Spencer morphism" constructed in [Mzk1], Chapter IX, is indeed a natural arithmetic analogue of the usual geometric Kodaira-Spencer morphism.* Indeed:

> *The comparison isomorphism of Theorem 6.2 renders explicit the sense in which the characters U^k on the "continuous torsion subgroup" $\mathbf{G}_{\mathrm{m}}$ of the elliptic curve define (through their role as "horizontal sections"), the usual Gauss-Manin connection on the (polarized) universal extension.*

Thus, in particular, this comparison isomorphism makes explicit the fact that looking at the extent to which the Hodge filtration is preserved by the Gauss-Manin connection on the "de Rham cohomology" (= in this case, the polarized universal extension of the elliptic curve) — i.e., the recipe for the usual geometric Kodaira-Spencer morphism — is essentially equivalent to looking at the extent to which the Hodge filtration is preserved by permutations of the characters on the torsion — i.e., the recipe for the "arithmetic Kodaira-Spencer morphism" of [Mzk1], Chapter IX.

One interesting consequence of Theorem 6.2 is that it implies that the *p-curvature of the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ vanishes identically* for all prime numbers p (cf. Corollaries 8.2, 8.3). This property is interesting in that it is somewhat different from what might expect, considering the behavior of more classical objects with connection such as $E^\dagger$ itself (cf. the discussion of §6 for more details). In fact, a stronger result holds: Namely, not only the p-curvature, but also the *"higher p-curvatures"* (introduced in [Mzk3], Chapter II, §2.1; and reviewed in §7.1 of the present paper) of the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ vanish identically for all prime numbers p (cf. Corollaries 8.2, 8.3). This property of the higher p-curvatures is of *essential importance* in the proof of the extendability (over the entire moduli stack of elliptic curves) of the étale integral structure on E^* (cf. Theorem 4.3; §8.3).

In §8, we study more of the *intrinsic properties* of the connection constructed in §5. One of the most fundamental such properties is that, unlike more classical objects with connection for which the connection gives rise to jumps of length ≤ 1 in the Hodge filtration ("Griffiths transversality"), the connection of §5 on the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ gives rise to jumps of length ≤ 2 on the Hodge filtration. We refer to this property as *"Griffiths semi-transversality."* Using this property, one can define a certain analogue for the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ of the classical *Kodaira-Spencer morphism* of a family of elliptic curves. In §8.1, we compute the Kodaira-Spencer morphism explicitly and show that it is equal to precisely $\frac{1}{2}$ of the classical Kodaira-Spencer morphism (Theorem 8.1). In §8.3, we define analogues for the pair $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ of the classical *Hasse invariant* of a family of elliptic curves in characteristic p. Moreover, whereas the classical Hasse invariants only involve the p-curvature, in §8.3, we consider "higher analogues" of this sort of invariant, involving the higher p-curvatures. Then, using a general formula (i.e., having nothing to do with the Hodge-Arakelov theory of elliptic curves) for computing higher p-curvatures which we develop in §8.2, *we compute the various analogues of the Hasse invariant of the pair* $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ (cf. Theorem 8.9). These computations are of crucial importance in the proof of the extendability (over the entire moduli stack of elliptic curves) of the étale integral structure on E^* given as the end of §8.3.

Finally, in §9, we clear up the confusion concerning the discussion of the étale integral structure in [Mzk1] by describing explicitly the relationship between the theory of the present paper and that of [Mzk1].

To conclude, we remark that since the space $\mathcal{V}_{\mathcal{L}} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{\mathrm{et}}})$ of global sections of $\mathcal{L}_{E^*_{\mathrm{et}}}$ over E^*_{et} admits a natural *connection*, as well as a natural *Hodge filtration*, it is natural to ask if it does not also admit some sort of natural *Frobenius action* (i.e., like the $\mathcal{MF}^{\nabla}$-objects of [Falt], §2). It is the desire of the author to address this question in more detail in a future paper. Once such a Frobenius action is defined, it is natural to study the resulting *Frobenius invariants* (cf. the case of $\mathcal{MF}^{\nabla}$-objects; [Falt], §2) over some sort of ring of p-adic periods such as "$\mathbf{B}_{\mathrm{crys}}$." In the present context, however, since instead of "Griffiths transversality," our object only satisfies "Griffiths semi-transversality," it is natural to expect that the "p-adic periods of $\mathbf{B}_{\mathrm{crys}}$" are likely not to be sufficient, i.e., one expects that in addition to the "p-adic analogue of $2\pi i$," which is a certain copy of $\mathbf{Z}_p(1)$ lying inside $\mathbf{B}_{\mathrm{crys}}$, we will also need the *square root of this p-adic $2\pi i$*. That is to say, it seems that in the case of Griffiths semi-transversality, it is natural to work over a sort of quadratic extension of $\mathbf{B}_{\mathrm{crys}}$ obtained from $\mathbf{B}_{\mathrm{crys}}$ by adjoining a square root of this p-adic $2\pi i$. This conforms to the idea that one expects this p-adic theory of "crystalline theta functions" to be a sort of p-adic analogue of the *complex Hodge theory of a polarized elliptic curve* (cf. [Mzk1], Chapter VII, §4), where square roots of 2π occur very naturally. We hope to address these issues in more detail in a future paper.

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§1. The Étale Integral Structure on the Universal Extension

In this §, we discuss the *étale integral structure on the universal extension of an elliptic curve.* This integral structure is discussed in [Mzk1], Chapter V, §3, but since certain technical details — especially concerning the *cohomology and global sections* of the structure sheaf of the universal extension equipped with the étale integral structure — were omitted in that discussion, we would like to discuss them in the following since they will be of substantial importance in the present paper.

Let $S^{\log}$ be a fine noetherian log scheme, and

$$C^{\log} \to S^{\log}$$

a *log elliptic curve* (cf. [Mzk1], Chapter III, Definition 1.1) over $S^{\log}$. Write $D \subseteq S$ for the pull-back to S of the divisor at infinity of the moduli stack of log elliptic curves, and $E \subseteq C$ for the one-dimensional semi-abelian scheme which forms an open subscheme of the semi-stable compactification C. Also, let us write

$$E^{\dagger} \to E$$

for the *universal extension* of E (cf. [Mzk1], Chapter III, Definition 1.2). We shall write ω_E for the line bundle on S given by the relative cotangent bundle of E over S restricted to the zero section $0_E : S \to E$. Then $E^{\dagger}$ has a natural structure of ω_E*-torsor* over E. Alternatively, one may think of $E^{\dagger}$ as a (commutative) *group scheme* over S which surjects (in the category of group schemes) onto E, with kernel given by the S-group scheme W_E defined by the line bundle ω_E.

Next, let us write $\widehat{S}$ for the formal scheme given by completing S along D. Let us denote the result of base-changing objects over S to $\widehat{S}$ by means of a subscript $\widehat{S}$. Then one has a natural identification

$$E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$$

Moreover, $E^{\dagger}_{\widehat{S}} \to E_{\widehat{S}}$ admits a *unique splitting*

$$\kappa_{\widehat{S}} : E_{\widehat{S}} \to E^{\dagger}_{\widehat{S}}$$

(cf. [Mzk1], Chapter III, Theorem 2.1) in the category of group objects over $\widehat{S}$. Moreover, over $\widehat{S}$, the line bundle ω_E admits a natural trivialization defined by the differential $d\,\log(U) = dU/U$ on $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}} = E_{\widehat{S}}$ (where U is the standard multiplicative coordinate on $\mathbf{G}_{\mathrm{m}}$). Thus, in particular, by using $\kappa_{\widehat{S}}$ and $d\,\log(U)$, one may think of the push-forward $\mathcal{R}$ of the structure sheaf $\mathcal{O}_{E^{\dagger}_{\widehat{S}}}$ to $E_{\widehat{S}}$ as being given by a *polynomial algebra*:

$$\mathcal{R} = \mathcal{O}_{E_{\widehat{S}}}[T]$$

(where the indeterminate T is that defined by the trivialization of ω_E given by $d\log(U)$).

In [Mzk1], Chapter III, §6, we defined (in the case where S is, say, $\mathbf{Z}$-*flat*) a new integral structure on this algebra $\mathcal{R}$ — which we shall refer to as the *étale integral structure* — as follows: First, let us write (for $r \in \mathbf{Z}_{\geq 0}$)

$$T^{[r]} \stackrel{\text{def}}{=} \binom{T}{r} = \frac{1}{r!}T(T-1)(T-2)\cdot \ldots \cdot (T-(r-1)) \in \mathcal{R}_{\mathbf{Q}} \stackrel{\text{def}}{=} \mathcal{R} \otimes \mathbf{Q}$$

Thus, if we define the operator δ on polynomials $f \in \mathcal{O}_S[T]$ by:

$$\delta(f) \stackrel{\text{def}}{=} f(T+1) - f(T) \in \mathcal{R}$$

then $\delta(T^{[r]}) = T^{[r-1]}$. Then the *étale integral structure* on $\mathcal{R}_{\mathbf{Q}}$ is given by:

$$\mathcal{R}^{\text{et}} \stackrel{\text{def}}{=} \bigoplus_{r \geq 0} \mathcal{O}_{E_{\widehat{S}}} \cdot T^{[r]}$$

It follows immediately from Lemma 1.1 below that this integral structure respects the $\mathcal{O}_{E_{\widehat{S}}}$-algebra structure of $\mathcal{R}$, as well as the structure of Hopf algebra on $\mathcal{R}$ arising from the fact that $E^{\dagger}$ is *group scheme over* S.

Lemma 1.1. *For nonnegative integers r_1, r_2, we have:*

(i) $T^{[r_1]} \cdot T^{[r_2]}$ is a $\mathbf{Z}$-linear combination of the $T^{[j]}$, for $j \in \mathbf{Z}_{j \geq 0}$.

(ii) $\binom{T_1+T_2}{r} = \sum_{j=0}^{r} \binom{T_1}{j} \cdot \binom{T_2}{r-j}$ (where T_1, T_2 are indeterminates).

Proof. Property (i) follows immediately from the well-known fact that the $\mathbf{Z}$-linear span of the $T^{[j]}$ (for $j \in \mathbf{Z}_{\geq 0}$) may be identified with the *ring* of $\mathbf{Z}$-valued polynomial functions on $\mathbf{Z}$. Property (ii) follows by considering the formal identity $(1+x)^{T_1+T_2} = (1+x)^{T_1} \cdot (1+x)^{T_2}$ (cf. [Mzk1], Chapter III, Lemma 7.5). ◯

Remark. The name "étale integral structure" arises from the fact that although in origin, the universal extension $E^{\dagger}$ is a "de Rham-theoretic object," its integral structure is not well-suited to *restriction to torsion points* (which are "étale-theoretic"). The "étale integral structure" is thus the natural integral structure on $E^\dagger$ which is compatible with the integral structure arising from restriction to torsion points (cf. [Mzk1], Chapter V, §3).

Remark. The notation $T^{[r]}$ is reminiscent of the notation typically used in discussions of the crystalline site for the *divided powers* $\frac{1}{r!}T^r$. The point of view that the author wishes to express here is the following: Just as divided powers arise naturally in discussions of "usual *continuous* calculus" and satisfy such properties as $\frac{d}{dT}(\frac{1}{r!}T^r) = \frac{1}{(r-1)!}T^{(r-1)}$ (with respect to "continuous differentiation"), the $T^{[r]}$'s satisfy $\delta(T^{[r]}) = T^{[r-1]}$ (where we wish to think of δ as a sort of "discrete derivative") and are naturally adapted to discussions of "*discrete* calculus" (cf., e.g., [Mzk1], Introduction, §3.4). We would thus like to refer to them as *discrete divided powers.*

In [Mzk1], Chapter V, §3, the issue of *extending the étale integral structure over all of S* is discussed. This issue of extending the étale integral structure is closely related to the issue of extending the section $\kappa_{\widehat{S}}$. Let us assume just in the following discussion that S is an *affine scheme* which is *étale* (i.e., the classifying morphism defined by $E \to S$ is étale) over the moduli stack $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ of (*smooth!*) elliptic curves over $\mathbf{Z}_p$ (for some prime number p). Let us write

$$H \to E$$

for the isogeny given by multiplication by p^n. Thus, $H \to S$ is another copy of $E \to S$.

It was shown in [Mzk1], Chapter V, §3, that $E^\dagger \to E$ admits a *natural section* modulo p^n over H. In the following, we give a slightly different construction of this section from that given in *loc. cit.* First, let us consider the pull-back morphisms

$$H^1(E, \omega_{E/S}) \xrightarrow{\phi} H^1(H, \omega_{E/S}|_H) \xrightarrow{\psi} H^1(H, \omega_{H/S})$$

induced by the isogeny $H \to E$. Note that all three of these cohomology modules are projective $\mathcal{O}_S$-modules of rank 1. In fact, the first and third cohomology modules even have natural trivializations given by the so called "residue map," which (by elementary algebraic geometry) may also be thought of as the "degree." Thus, relative to these trivializations, the composite $\psi \circ \phi$ corresponds to multiplication by p^{2n} (i.e., the degree of the isogeny $H \to E$). On the other hand, by functoriality, the morphism $\omega_{E/S}|_H \to \omega_{H/S}$ is simply multiplication by p^n. Thus, in particular, we obtain that (for appropriate trivializations) the morphism $\phi : H^1(E, \omega_{E/S}) \to H^1(H, \omega_{E/S}|_H)$ may be identified with multiplication by $p^{2n}/p^n = p^n$. In particular, it follows that the ω_E-torsor $E^\dagger \to E$ *splits modulo p^n over H*. Moreover, since $H^0(H, \omega_{E/S}|_H) = \omega_E$ (i.e., such sections are constant on the fibers of $H \to S$), it follows that *restriction to the zero section $0_H : S \to H$ of H defines a natural equivalence of categories between splittings modulo p^n of $E^\dagger|_H$ and splittings modulo p^n of $E^\dagger|_{0_H} = E^\dagger|_{0_E}$*. On the other hand, $E^\dagger$ is a *group scheme*, hence is equipped with its own zero section $0_{E^\dagger}$ which thus defines a natural splitting of $E^\dagger|_{0_E} \to S$, hence defines a unique splitting modulo p^n of $E^\dagger|_H$, which we denote by:

$$\kappa_H : H_{\mathbf{Z}/p^n\mathbf{Z}} \to E^\dagger$$

(where the subscript $\mathbf{Z}/p^n\mathbf{Z}$ denotes base-change to $\mathbf{Z}/p^n\mathbf{Z}$). Note that since this splitting is *natural*, it agrees on fiber products (over $\mathcal{M}_{1,0}$) of different S's, hence is defined over the moduli stack $(\mathcal{M}_{1,0})_{\mathbf{Z}/p^n\mathbf{Z}}$. Moreover, κ_H is *compatible* with the splitting $\kappa_{\widehat{S}}$ in the following sense: When the base S is taken to be complement of the point at infinity in a neighborhood of infinity of $\overline{\mathcal{M}}_{1,0}$ — i.e., $S = \mathrm{Spec}((\mathbf{Z}/p^n\mathbf{Z})[[q]][q^{-1}])$ (where q is the "*q-parameter*," defined in a formal neighborhood of the point at infinity of $\overline{\mathcal{M}}_{1,0}$) — then κ_H *coincides with the section of* $E^\dagger|_H$ *defined by* $\kappa_{\widehat{S}}$ (cf. [Mzk1], Chapter V, §3, for more details). (Indeed, this follows from the above discussion and the fact that both sections pass through the zero section $0_{E^\dagger}$ of $E^\dagger$.)

The section κ_H is the *key ingredient* that allows one to extend the étale integral structure over all of $\overline{\mathcal{M}}_{1,0}$, as desired. This fact is observed in [Mzk1], Chapter V, §3 (at the end of the "Analytic Continuation Argument"), but the technicalities supporting this observation were regrettably omitted in *loc. cit.* (a fact for which the author wishes to apologize to readers of [Mzk1]), so we would like to discuss them in the following (since these details turn out to be important in the general framework of the present paper).

Let us return to discussing the situation for a general S for which S is $\mathbf{Z}$-*flat*. Also, since we have already defined the étale integral structure in a neighborhood of infinity, let us assume that $E \to S$ is *proper* (so $S^{\log}$ is equipped with the trivial log structure). Write $\mathcal{R}$ for the push-forward of the structure $\mathcal{O}_{E^\dagger}$ to E. Recall that $\mathcal{R}$ is equipped with a *natural filtration*

$$F^r(\mathcal{R}) \subseteq \mathcal{R}$$

given by the functions whose *torsorial degree* (i.e., degree when thought of as polynomials in the relative affine variable of $E^\dagger \to E$) $< r$. Thus, $F^r(\mathcal{R})$ is a *vector bundle on S of rank r*. Moreover, we have a *natural identification*

$$(F^{r+1}/F^r)(\mathcal{R}) = \tau_E^{\otimes r}|_E$$

for $r \geq 0$ (where τ_E is the dual bundle to ω_E).

Now we would like to construct *new integral structures* $F^r(\mathcal{R}^{\text{et}})$ on $F^r(\mathcal{R})$ which extend the étale integral structure in a neighborhood of infinity. (Here, when we speak of "integral structures" $F^r(\mathcal{R}^{\text{et}})$, we will always assume that $F^r(\mathcal{R}) \subseteq F^r(\mathcal{R}^{\text{et}})$.) We would like to do this by *induction on r*. Consider the following conditions on $F^r(\mathcal{R}^{\text{et}})$:

(I_r) The integral structure induced on $(F^{r+1}/F^r)(\mathcal{R}) = \tau_E^{\otimes r}|_E$ by $F^r(\mathcal{R}^{\text{et}})$ is given by $\frac{1}{r!} \cdot \tau_E^{\otimes r}|_E$.

(II_r) The surjection $F^r(\mathcal{R}^{\text{et}}) \to \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_E$ (cf. (I_{r-1})) induces an isomorphism on cohomology after arbitrary base-change:

$$\mathbf{R}f^1_*(F^r(\mathcal{R}^{\text{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) \cong \mathbf{R}f^1_*(\frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_E \otimes_{\mathcal{O}_S} \mathcal{O}_T)$$

(where $f : E \to S$ is the structure morphism, and $T \to S$ is an *arbitrary* — i.e., not necessarily $\mathbf{Z}$-flat — S-scheme).

(III_r) For any S-scheme T, we have $f_*(F^r(\mathcal{R}^{\text{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \mathcal{O}_T$.

(IV_r) Write $H \to E$ for the isogeny given by multiplication by p^n (so H is just another copy of E). Then the morphism $F^r(\mathcal{R}) \to \mathcal{O}_H \otimes \mathbf{Z}/p^n\mathbf{Z}$ induced by the section κ_H constructed above factors through $F^r(\mathcal{R}^{\text{et}})$.

(V_r) When $S = \text{Spec}(\mathbf{Z}[[q]][q^{-1}])$ (where q is the "*q-parameter*," defined in a formal neighborhood of the point at infinity of $\overline{\mathcal{M}}_{1,0}$), the integral structure $F^r(\mathcal{R}^{\text{et}})$ is that given by the "$\mathcal{R}^{\text{et}} \stackrel{\text{def}}{=} \bigoplus_{j \geq 0} \mathcal{O}_{E_{\widehat{S}}} \cdot T^{[j]}$" discussed above (cf. [Mzk1], Chapter V, §3, for more details).

We would like to construct $F^r(\mathcal{R}^{\text{et}})$ by induction on r in such a way that it is functorial in ($\mathbf{Z}$-flat) S and, moreover, satisfies the above five properties.

Let us observe first that taking $F^1(\mathcal{R}^{\text{et}}) \stackrel{\text{def}}{=} F^1(\mathcal{R}) = \mathcal{O}_E$ satisfies the above properties and is functorial in S. Now let us suppose that $F^r(\mathcal{R}^{\text{et}})$ has been constructed for some $r \geq 1$ (which satisfies the above five properties and is functorial in S). Then $F^{r+1}(\mathcal{R}^{\text{et}})$ may be constructed as follows. First, let us push-forward the exact sequence

$$0 \to F^r(\mathcal{R}) \to F^{r+1}(\mathcal{R}) \to \tau_E^{\otimes r}|_E \to 0$$

by $F^r(\mathcal{R}) \hookrightarrow F^r(\mathcal{R}^{\text{et}})$ to obtain an exact sequence:

$$0 \to F^r(\mathcal{R}^{\text{et}}) \to F^{r+1}(\mathcal{R})' \to \tau_E^{\otimes r}|_E \to 0$$

If we then consider the associated long exact sequence in cohomology over the base T (where T is an S-scheme), we obtain:

$$\begin{aligned}
&0 \to f_*(F^r(\mathcal{R}^{\text{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \mathcal{O}_T \to f_*(F^{r+1}(\mathcal{R})' \otimes_{\mathcal{O}_S} \mathcal{O}_T) \to \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T \\
&\stackrel{\partial'}{\to} \mathbf{R}^1 f_*(F^r(\mathcal{R}^{\text{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathbf{R}^1 f_*(F^{r+1}(\mathcal{R})' \otimes_{\mathcal{O}_S} \mathcal{O}_T) \\
&\to \tau_E^{\otimes r+1}|_E \otimes_{\mathcal{O}_S} \mathcal{O}_T \to 0
\end{aligned}$$

(where we use properties (I_r), (II_r), and (III_r), together with the natural identification (by Serre duality) of $\mathbf{R}^1 f_*(\mathcal{O}_E)$ with τ_E).

Lemma 1.2. *The morphism* $\partial' : \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T$ *is given by multiplication by* r *on* $\tau_E^{\otimes r}$*, followed by the natural inclusion* $\tau_E^{\otimes r} \hookrightarrow \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r}$ *(tensored over* $\mathcal{O}_S$ *with* $\mathcal{O}_T$*).*

Proof. The connecting morphism $\tau_E \to \mathbf{R}^1 f_*(\mathcal{O}_E)$ for the exact sequence

$$0 \to F^1(\mathcal{R}) = \mathcal{O}_E \to F^2(\mathcal{R}) \to \tau_E|E \to 0$$

may be identified (up to perhaps a sign — which is irrelevant here) with the identification $\tau_E = \mathbf{R}^1 f_*(\mathcal{O}_E)$ (via Serre duality referred to above) — cf. [Mzk1], Chapter III, Theorem 4.2. It thus follows immediately from the "combinatorics of the symmetric algebra" — i.e., the fact that the automorphism of the polynomial algebra in an indeterminate X that maps $X \mapsto X + c$ (for some constant c) will necessarily map $X^r \mapsto X^r + r \cdot c \cdot X^{r-1} + \dots$ — that the connecting morphism $\tau_E^{\otimes r} \to \mathbf{R}^1 f_*(\tau_E^{\otimes r-1}) = \tau_E^{\otimes r}$ for the exact sequence

$$0 \to (F^r/F^{r-1})(\mathcal{R}) = \tau_E^{\otimes r-1} \to (F^{r+1}/F^{r-1})(\mathcal{R}) \to (F^{r+1}/F^r)(\mathcal{R}) = \tau_E^{\otimes r}|E \to 0$$

is given by multiplication by r. Since this connecting morphism is related to the connecting morphism ∂' in question by an obvious commutative diagram (which we leave to the reader to write out), the lemma follows immediately. ◯

Now suppose that $T \stackrel{\mathrm{def}}{=} S_{\mathbf{Z}/p^n\mathbf{Z}}$, where $n > a \stackrel{\mathrm{def}}{=} \mathrm{ord}_p(r!)$; write $b \stackrel{\mathrm{def}}{=} n - a$. Then we get an exact sequence

$$0 \to \mathcal{O}_T \to f_*(F^{r+1}(\mathcal{R})' \otimes_{\mathcal{O}_S} \mathcal{O}_T) \to \tau_E^{\otimes r} \otimes (p^b\mathbf{Z}/p^n\mathbf{Z}) \to 0$$

Note, moreover, that this exact sequence *splits*. Indeed, by property (IV_r), it follows that κ_H induces a morphism $F^{r+1}(\mathcal{R})' \to \mathcal{O}_H \otimes \mathbf{Z}/p^n\mathbf{Z}$; applying f_* to this morphism thus induces a splitting α of the above exact sequence. Moreover, the above exact sequence induces an isomorphism:

$$\mathrm{Ker}(\alpha) \cong \tau_E^{\otimes r} \otimes (p^b\mathbf{Z}/p^n\mathbf{Z})$$

Now observe that this splitting α *induces a splitting* of the surjection

$$F^{r+1}(\mathcal{R})' \otimes (p^b\mathbf{Z}/p^n\mathbf{Z}) \to \tau_E^{\otimes r}|_E \otimes (p^b\mathbf{Z}/p^n\mathbf{Z})$$

of vector bundles on $E \otimes (\mathbf{Z}/p^a\mathbf{Z})$. Let us denote the image of $\tau_E^{\otimes r}|_E \otimes (p^b\mathbf{Z}/p^n\mathbf{Z})$ via this splitting by

$$\mathcal{M} \subseteq F^{r+1}(\mathcal{R})' \otimes (p^b\mathbf{Z}/p^n\mathbf{Z})$$

Next, *let us define*

$$F^{r+1}(\mathcal{R}^{\mathrm{et}}) \subseteq \frac{1}{r!} \cdot F^{r+1}(\mathcal{R})'$$

at the prime p as the subsheaf generated by the sections m of $\frac{1}{r!} \cdot F^{r+1}(\mathcal{R})$ for which $p^n \cdot m \in \mathcal{M}$. (Here, we think of multiplication by p^n as defining an isomorphism $F^{r+1}(\mathcal{R})' \otimes (p^{-a}\mathbf{Z}/\mathbf{Z}) \cong F^{r+1}(\mathcal{R})' \otimes (p^b\mathbf{Z}/p^n\mathbf{Z})$.)

Then one checks easily that this definition is *independent of* n, and that the resulting $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ is a vector bundle on $\mathcal{O}_E$ of rank $r+1$ which contains $F^r(\mathcal{R}^{\mathrm{et}})$ (in such a way that $(F^{r+1}/F^r(\mathcal{R}^{\mathrm{et}})$ is *torsion free*) and satisfies the condition (I_{r+1}) discussed above. Thus, we get a long exact sequence (for an arbitrary S-scheme T):

$$\begin{aligned} 0 \to \mathcal{O}_T \stackrel{\beta}{\to}\ & f_*(F^{r+1}(\mathcal{R}^{\mathrm{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) \to \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T \\ \stackrel{\partial^{\mathrm{et}}}{\to}\ & \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathbf{R}^1 f_*(F^{r+1}(\mathcal{R}^{\mathrm{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) \\ \stackrel{\gamma}{\to}\ & \frac{1}{r!} \cdot \tau_E^{\otimes r+1}|_E \otimes_{\mathcal{O}_S} \mathcal{O}_T \to 0 \end{aligned}$$

(cf. the exact sequence discussed above in the case of $F^{r+1}(\mathcal{R})'$). Moreover, since the connecting morphism ∂^{et} is given by multiplication by r (cf. Lemma 1.2), it follows that ∂^{et} is an *isomorphism.* In particular, we obtain that β and γ are also isomorphisms, so conditions (II_{r+1}) and (III_{r+1}) are also satisfied by $F^{r+1}(\mathcal{R}^{\mathrm{et}})$. Moreover, it follows from the definition of the section α and the construction of $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ that $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ also satisfies (IV_{r+1}). To see that $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ satisfies (V_{r+1}), it suffices to observe that, for instance, (I_{r+1}) and (IV_{r+1}) uniquely determine $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ (cf. the construction of $F^{r+1}(\mathcal{R}^{\mathrm{et}})$ by means of α), and that the "$F^{r+1}(\mathcal{R}^{\mathrm{et}})$" defined above in a neighborhood of infinity also satisfies (I_{r+1}) and (IV_{r+1}) (indeed, this essentially amounts to the fact that the $T^{[j]}$ assume *integral values* on all elements of $\mathbf{Z}$ — cf. [Mzk1], Chapter V, §3, for more details). This completes the construction of the $F^r(\mathcal{R}^{\mathrm{et}})$ (for all $r \geq 1$).

We summarize the above discussion as follows:

Theorem 1.3. (Étale Integral Structure on the Universal Extension) *Let $S^{\log}$ be a fine, $\mathbf{Z}$-flat noetherian log scheme. Let $C^{\log} \to S^{\log}$ be a log elliptic curve. Write $E \subseteq C$ for the open subscheme which is a one-dimensional semi-abelian scheme, and $E^\dagger \to E$ for the universal extension. Thus, there is a natural ω_E-torsor $E_C^\dagger \to C$ that extends $E^\dagger \to E$*

(cf. [Mzk1], Chapter III, Corollary 4.3). Denote by $\mathcal{R}$ the push-forward of $\mathcal{O}_{E_C^\dagger}$ to C, equipped with its natural filtration $F^r(\mathcal{R})$ (given by considering sections whose torsorial degree is $< r$). Then there is a **natural integral structure** *(functorial in $S^{\log}$)*

$$\mathcal{R}^{\text{et}} \subseteq \mathcal{R}_{\mathbf{Q}}$$

that satisfies the following properties:

- **(I) (Subquotients)** *The integral structure induced on $(F^{r+1}/F^r)(\mathcal{R}) = \tau_E^{\otimes r}|_C$ by $F^r(\mathcal{R}^{\text{et}})$ is given by $\frac{1}{r!} \cdot \tau_E^{\otimes r}|_C$.*

- **(II) (Cohomology)** *The surjection $F^r(\mathcal{R}^{\text{et}}) \to \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_C$ (cf. (I)) induces an isomorphism on cohomology after arbitrary base-change:*

$$\mathbf{R}f^1_*(F^r(\mathcal{R}^{\text{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) \cong \mathbf{R}f^1_*(\frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_C \otimes_{\mathcal{O}_S} \mathcal{O}_T)$$

  (where $f : C \to S$ is the structure morphism, and $T \to S$ is an arbitrary — i.e., not necessarily $\mathbf{Z}$-flat — S-scheme). In particular, we have: $\mathbf{R}f^1_*(\mathcal{R}^{\text{et}} \otimes_{\mathcal{O}_S} \mathcal{O}_T) = 0$.

- **(III) (Global Sections)** *We have: $f_*(\mathcal{R}^{\text{et}} \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \mathcal{O}_T$ (for any S-scheme T).*

- **(IV) (Compatibility with the Canonical Section)** *Suppose that $E = C$. Write $H \to E$ for the isogeny given by multiplication by p^n (so H is just another copy of E). Then the morphism $F^r(\mathcal{R}) \to \mathcal{O}_H \otimes \mathbf{Z}/p^n\mathbf{Z}$ induced by the section κ_H constructed above factors through $F^r(\mathcal{R}^{\text{et}})$.*

- **(V) (Description at Infinity)** *When $S = \mathrm{Spec}(\mathbf{Z}[[q]][q^{-1}])$ (where q is the "q-parameter," defined in a formal neighborhood of the point at infinity of $\overline{\mathcal{M}}_{1,0}$), the integral structure $F^r(\mathcal{R}^{\text{et}})$ is that given by the "$\mathcal{R}^{\text{et}} \stackrel{\text{def}}{=} \bigoplus_{j \geq 0} \mathcal{O}_{E_{\widehat{S}}} \cdot T^{[j]}$" discussed above (cf. [Mzk1], Chapter V, §3, for more details).*

- **(VI) (Group Scheme Structure)** *$\mathcal{R}^{\text{et}}$ is an $\mathcal{O}_C$-algebra which is compatible with the group scheme structure of $E^\dagger$.*

Thus, in particular, $E^\dagger_{\text{et}} \stackrel{\text{def}}{=} Spec(\mathcal{R}^{\text{et}}|_E)$ defines a **group scheme** *(which is not of finite type) over S equipped with a homomorphism*

$$E^\dagger_{\text{et}} \to E$$

Moreover, over $\mathbf{Q}$, this homomorphism may be identified with $E^\dagger \to E$.

Proof. It remains only to verify property (VI). But this follows immediately from property (V) and the discussion preceding Lemma 1.1 above. (Here we use the fact that the moduli stack $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is regular of dimension 2, so integral structures are completely determined once they are determined over $\mathbf{Q}$ and near the point at infinity.) $\bigcirc$

Remark. The observation that property (VI) (of Theorem 1.3) holds arose from discussions between the author and A. Ogus in October 1999.

Remark. Properties (II) and (III) of Theorem 1.3 will be of *fundamental importance* in this paper. Note that these properties do not hold for the universal extension with its usual integral structure.

§2. The Étale Integral Structure for an Ordinary Elliptic Curve

Let p be a prime number. In this §, we make *explicit* the structure of the p-adic completion $(E^{\dagger}_{\mathrm{et}})^{\wedge}$ of the universal extension equipped with the étale integral structure, in the case of an elliptic curve whose reduction modulo p is *ordinary.* In particular, we construct an isomorphism

$$(E^{\dagger}_{\mathrm{et}})^{\wedge} \cong E^{\mathrm{F}^{\infty}}$$

between this p-adic completion and a certain object $E^{\mathrm{F}^{\infty}}$ obtained by considering composites of the "Verschiebung morphism associated to an ordinary elliptic curve."

§2.1. Some p-adic Function Theory:

In the following discussion, if X and Y are *topological spaces*, then let us write

$$\mathrm{Cont}(X, Y)$$

for the space of *continuous functions* from X to Y. On the other hand, let us write

$$\mathrm{Comb}(\mathbf{Z}_p)$$

for the free $\mathbf{Z}_p$-module generated by the symbols $T^{[j]}$, for $j \in \mathbf{Z}_{\geq 0}$. Thus, by thinking of the symbol $T^{[j]}$ as the continuous $\mathbf{Z}_p$-valued function on $\mathbf{Z}_p$ that maps $\lambda \in \mathbf{Z}_p$ to $\binom{\lambda}{j}$, we obtain a natural morphism $\mathrm{Comb}(\mathbf{Z}_p) \rightarrow \mathrm{Cont}(\mathbf{Z}_p, \mathbf{Z}_p)$. One sees easily that this morphism extends naturally to the p-adic completion $\mathrm{Comb}(\mathbf{Z}_p)^\wedge$ of $\mathrm{Comb}(\mathbf{Z}_p)$. Moreover, it is well-known (by a result of Mahler — cf., e.g., [Katz3], §3.2) that:

Lemma 2.1. *The resulting morphism*

$$\mathrm{Comb}(\mathbf{Z}_p)^\wedge \to \mathrm{Cont}(\mathbf{Z}_p, \mathbf{Z}_p)$$

is a bijection.

§2.2. The Verschiebung Morphism:

Let S be a *p-adic formal scheme S which is formally smooth over $\mathbf{Z}_p$*. Assume also that we are given a *family of ordinary elliptic curves $E \to S$* such that the associated classifying morphism $S \to (\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ is formally (i.e., relative to the p-adic topology) *étale*. (Here, by "ordinary," we mean that the fibers of $E \to S$ over all the points of $S_{\mathbf{F}_p}$ have nonzero Hasse invariant.) For $n \geq 1$, write

$$E[p^n] \stackrel{\mathrm{def}}{=} \mathrm{Ker}([p^n] : E \to E)$$

for the kernel of multiplication by p^n on E. Then, as is well-known (cf., e.g., [Katz4], p. 150), there is a unique exact sequence

$$0 \to E[p^n]^{\boldsymbol{\mu}} \to E[p^n] \to E[p^n]^{\mathrm{et}} \to 0$$

of finite flat group schemes over E such that $E[p^n]^{\boldsymbol{\mu}}$ (respectively, $E[p^n]^{\mathrm{et}}$) is étale locally isomorphic to $\boldsymbol{\mu}_{p^n}$ (respectively, $\mathbf{Z}/p^n\mathbf{Z}$).

Let us write

$$E^{\mathrm{F}^n} \stackrel{\mathrm{def}}{=} E/E[p^n]^{\boldsymbol{\mu}}$$

Then since $E^{\mathrm{F}^n} \to S$ is a family of elliptic curves, and the classifying morphism associated to $E \to S$ is *étale*, it follows that $E^{\mathrm{F}^n} \to S$ defines a morphism

$$\Phi_S^n : S \to S$$

One checks easily that $\Phi_S \stackrel{\mathrm{def}}{=} \Phi_S^1$ is a *lifting of the Frobenius morphism in characteristic* p, and that Φ_S^n is the result of iterating Φ_S a total of n times (as the notation suggests). The morphism

$$\mathcal{V} : E^{\mathrm{F}} \to E$$

given by dividing E^{F} by the image in E^{F} of $E[p]$ will be referred to as the the *Verschiebung morphism* associated to E. For any $n \geq 1$, the morphism

$$\mathcal{V}^n : E^{\mathrm{F}^n} \to E$$

given by dividing E^{F^n} by the image in E^{F^n} of $E[p^n]$ is easily seen to be equal (as the notation suggests) to the "n-th iterate" (i.e., up to various appropriate base changes by iterates of Φ_S) of $\mathcal{V}$. Note that the kernel of $\mathcal{V}^n$ may be identified with $E[p^n]^{\mathrm{et}}$. In particular, it follows that $\mathcal{V}^n$ is *étale of degree* p^n.

Thus, we obtain a tower

$$\ldots \to E^{\mathrm{F}^n} \to E^{\mathrm{F}^{n-1}} \to \ldots \to E^{\mathrm{F}} \to E$$

of étale isogenies of degree p. Let us denote the p-adic completion of the inverse limit of this system of isogenies by E^{F^∞}. Thus, in particular, we have a natural morphism

$$E^{\mathrm{F}^\infty} \to E$$

The goal of the present § is to construct a natural isomorphism

$$E^{\mathrm{F}^\infty} \cong (E^\dagger_{\mathrm{et}})^\wedge$$

between E^{F^∞} *and the p-adic completion of* $E^\dagger_{\mathrm{et}}$.

We begin by constructing a morphism

$$\kappa[p^n] : E^{\mathrm{F}^n} \otimes \mathbf{Z}/p^n\mathbf{Z} \to E^\dagger \otimes \mathbf{Z}/p^n\mathbf{Z}$$

as follows: First, observe that since $\mathcal{V}^n : E^{\mathrm{F}^n} \to E$ is finite étale of degree p^n, it follows that the pull-back via $\mathcal{V}^n$ of any ω_E-torsor on E *splits modulo* p^n (cf. the argument involving the isogeny "$H \to E$" in §1). Thus, if we take the (unique!) splitting modulo p^n of the pull-back of $E^\dagger \to E$ via $\mathcal{V}^n$ which maps the origin of E^{F^n} to that of $E^\dagger$, we obtain a morphism $\kappa[p^n] : E^{\mathrm{F}^n} \otimes \mathbf{Z}/p^n\mathbf{Z} \to E^\dagger \otimes \mathbf{Z}/p^n\mathbf{Z}$, as desired. Note, moreover, that since the morphism $H \to E$ (given by multiplication by p^n) considered in §1 factors through E^{F^n}, it makes sense to compare the splitting $\kappa_H \otimes \mathbf{Z}/p^n\mathbf{Z}$ of §1 with $\kappa[p^n]|_H$. Then it follows immediately from the definitions that *these two splittings coincide.*

Now, if we pass to the limit $n \to \infty$, we obtain a morphism

$$\kappa[p^\infty] : E^{\mathrm{F}^\infty} \to (E^\dagger)^\wedge$$

(where "$\wedge$" denotes p-adic completion). Finally, by the property of Theorem 1.3, (IV) (i.e., compatibility of the étale integral structure with κ_H), together with the coincidence "$\kappa_H \otimes \mathbf{Z}/p^n\mathbf{Z} = \kappa[p^n]|_H$" observed above (and the fact that the morphisms $H \to E^{\mathrm{F}^n} \to E$ in the factorization of $H \to E$ are both faithfully flat), we obtain that $\kappa[p^\infty]$ factors through the étale integral structure of $E^\dagger$, i.e., we obtain a morphism

$$\kappa_{\mathrm{et}}^\infty : E^{\mathrm{F}^\infty} \to (E_{\mathrm{et}}^\dagger)^\wedge$$

as desired.

Theorem 2.2. (Explicit Description of the Étale Integral Structure of an Ordinary Elliptic Curve) *The natural morphism*

$$\kappa_{\mathrm{et}}^\infty : E^{\mathrm{F}^\infty} \to (E_{\mathrm{et}}^\dagger)^\wedge$$

from the p-adic completion of the "Verschiebung tower" of E to the p-adic completion of the universal extension of E equipped with the étale integral structure is an isomorphism.

Proof. Write $G \stackrel{\mathrm{def}}{=} E^{\mathrm{F}^\infty}$. Thus, we have a natural morphism $G \to E$. In order to prove that $\kappa_{\mathrm{et}}^\infty$ is an isomorphism, we would like to regard $\kappa_{\mathrm{et}}^\infty$ and $\kappa[p^\infty]$ as *morphisms over* E; then to base-change these morphisms by the faithfully flat morphism $G \to E$; and finally, to apply Lemma 2.1 to show that the result of base-changing $\kappa_{\mathrm{et}}^\infty$ by $G \to E$ is an isomorphism. This will complete the proof that $\kappa_{\mathrm{et}}^\infty$ itself is an isomorphism.

Write E_{et} for the inverse limit of the $E[p^n]^{\mathrm{et}}$ via the natural projection morphisms. Thus, $E_{\mathrm{et}} \to S$ is a "profinite étale covering." Moreover, since $E^{\mathrm{F}^\infty} \to E$ admits a natural structure of E_{et}-covering, it follows that we have a natural isomorphism

$$E^{\mathrm{F}^\infty} \times_E G \cong E_{\mathrm{et}} \times_S G$$

On the other hand, if we base-change $(E^\dagger)^\wedge \to E$ by $G \to E$, then the resulting $(E^\dagger)^\wedge|_G \to G$ admits a *splitting* (defined by $\kappa[p^\infty]$)

$$(E^\dagger)^\wedge \times_E G \cong (\mathbf{V}(\omega_E))^\wedge \times_S G$$

(where $\mathbf{V}(-)$ denotes the affine group scheme over S defined by the line bundle in parentheses). Moreover, let us observe that if we think of E_{et} as a local system of $\mathbf{Z}_p$-modules on S, then we have a natural isomorphism

$$E_{\mathrm{et}} \otimes_{\mathbf{Z}_p} \mathcal{O}_S \cong \omega_E$$

(cf., e.g., [Katz4], p. 163). Now I *claim* that relative to these natural isomorphisms, the morphism $\kappa[p^\infty]|_G$ amounts to the *natural inclusion*

$$E_{\mathrm{et}}|_G \hookrightarrow \mathbf{V}(E_{\mathrm{et}} \otimes_{\mathbf{Z}_p} \mathcal{O}_S)^\wedge|_G \cong (\mathbf{V}(\omega_E))^\wedge|_G$$

Indeed, this claim may be verified in a formal neighborhood of infinity, where it follows from the well-known structure of the universal extension in a formal neighborhood of infinity, as reviewed, for instance, in §1.

The elucidation of the structure of $\kappa[p^\infty]|_G$ in the preceding paragraph thus shows that (up to choosing a trivialization $E_{\mathrm{et}} \cong \mathbf{Z}_p$ over some profinite étale covering of S) the morphism induced by $\kappa_{\mathrm{et}}^\infty|_G$ on functions is none other than the morphism

$$\mathrm{Comb}(\mathbf{Z}_p)^\wedge \to \mathrm{Cont}(\mathbf{Z}_p, \mathbf{Z}_p)$$

considered in Lemma 2.1 (topologically tensored over $\mathbf{Z}_p$ with $\mathcal{O}_G$). Thus, Lemma 2.1 implies that $\kappa_{\mathrm{et}}^\infty|_G$, hence (by faithful flat descent) that $\kappa_{\mathrm{et}}^\infty$ is an isomorphism, as desired. ○

Remark. One way to think of the content of Theorem 2.2 is as the assertion that:

> *The étale integral structure on $E^\dagger$ is very closely related to the p-adic Hodge theory of E.*

Another manifestation of this phenomenon is the following: The splitting $\kappa_H : H_{\mathbf{Z}/p^n\mathbf{Z}} \to E^\dagger$ used in §1 to construct the étale integral structure on $E^\dagger$ defines a morphism

$$\kappa_H[p^n] : E[p^n] \to \mathbf{V}(\omega_E) \otimes \mathbf{Z}/p^n\mathbf{Z}$$

(i.e., by thinking of κ_H as a morphism over E and taking the morphism induced on fibers over the origin of E). Note that $\kappa_H[p^n]$ is defined *even without the ordinariness assumption on E*. Moreover, if we denote by $T \to S$ the normalization of S in the finite étale covering $T_{\mathbf{Q}_p} \to S_{\mathbf{Q}_p}$ given by considering p^n-level structures on E, and by $\mathcal{T}(E)$ the p-adic Tate module of E, then restricting $\kappa_H[p^n]$ to T defines a morphism

$$\alpha_n : \mathcal{T}(E) \otimes_{\mathbf{Z}_p} \mathcal{O}_T \otimes \mathbf{Z}/p^n\mathbf{Z} \to \omega_E|_T \otimes \mathbf{Z}/p^n\mathbf{Z}$$

which is the "fundamental morphism (often referred to as the 'period map') of the p-adic Hodge theory of E" (cf., e.g., the morphism "$\mathrm{Hom}(T_E, J_{\mathbf{Q}_p}) \to \omega_E \otimes_R \widehat{\overline{R}}_{\mathbf{Q}_p}$" of [Mzk1], Chapter IX, §2). Indeed, the coincidence of these two morphisms may be verified in a formal neighborhood of infinity, where it is clear from the discussion of [Mzk1], Chapter

IX, §2, together with the well-known structure of the universal extension (as reviewed, for instance, in §1 of the present paper).

Remark. Since $E^{\dagger}_{\mathrm{et}}$ is defined globally over $\mathbf{Z}$ whenever E is defined globally over $\mathbf{Z}$, Theorem 2.2 may also be taken as asserting that $E^{\dagger}_{\mathrm{et}}$ enjoys the following remarkable interpretation:

> *The universal extension of an elliptic curve equipped with the étale integral structure is a natural globalization over* $\mathbf{Z}$ *of the (very local!) p-adic Hodge theory of the Hodge elliptic curve.*

This interpretation is very much in line with the general philosophy of [Mzk1] and the present paper (cf., e.g., the Introduction of [Mzk1]).

§3. Compactified Hodge Torsors

In this §, we expand on the discussion of [Mzk1], Chapter III, §4, and, in particular, further clarify the *relationship between the universal extension of an elliptic curve and the Hodge-theoretic first Chern class of a line bundle on the elliptic curve.*

Let

$$C^{\log} \to S^{\log}$$

be a *log elliptic curve* as in §1. Also, we write $E \subseteq C$, $D \subseteq S$, $E^{\dagger} \to E$ (cf. §1) for the various objects associated to $C^{\log} \to S^{\log}$. In this §, we would also like to consider a *line bundle* $\mathcal{L}$ on C, whose relative degree over S we denote by d. Now define

$$C^{\mathcal{L}} \to C$$

to be the ω_E-torsor over C of *logarithmic connections* (relative to the morphism $C^{\log} \to S^{\log}$) *on the line bundle* $\mathcal{L}$. We will denote the open subscheme $C^{\mathcal{L}} \times_C E \subseteq C^{\mathcal{L}}$ by $E^{\mathcal{L}}$.

Next, we would like to show that $E^{\mathcal{L}}$ admits a natural structure of $E^{\dagger}$*-torsor.* First, let us observe that we have a natural *action* of $E^{\dagger}$ on $C^{\mathcal{L}}$. Indeed, suppose that we are given an S-scheme $T \to S$, a point $\alpha^{\dagger} \in E^{\dagger}(T)$ (whose image in $E(T)$ we denote by α), an open subscheme $U \subseteq C_T \stackrel{\mathrm{def}}{=} C \times_S T$, and a logarithmic connection ∇ on $\mathcal{L}|_U$. Write $U_{\alpha} \subseteq C_T$ (respectively, $\mathcal{L}_{\alpha}$; ∇_{α}) for the open subscheme $\subseteq C_T$ (respectively, line bundle on C_T; logarithmic connection on $\mathcal{L}_{\alpha}|_{U_{\alpha}}$) obtained from U (respectively, $\mathcal{L}$; ∇) by translating by α. Then *we must construct, from this data, a logarithmic connection* ∇^{α} *on* $\mathcal{L}|_{U_{\alpha}}$. To do this, we observe that the lifting $\alpha^\dagger$ of α corresponds to a logarithmic connection ∇' on $\mathcal{O}_{C_T}([\alpha]-[0_{C_T}])$ (where 0_{C_T} is the zero section of C_T). On the other hand, we have the following

Lemma 3.1. *We have an isomorphism of line bundles on C*

$$\mathcal{O}_{C_T}([\alpha]-[0_{C_T}])^{\otimes d} \cong \mathcal{L}_\alpha \otimes_{\mathcal{O}_{C_T}} \mathcal{L}_T^{-1} \otimes_{\mathcal{O}_T} \mathcal{N}$$

for some line bundle $\mathcal{N}$ on T.

Proof. First, let us observe that it suffices to show the existence of such an isomorphism *(Zariski) locally* on S. Moreover, after localizing on S, it follows from the fact that the divisor $[0_C] \subseteq C$ (defined by the zero section 0_C) is *ample* that $\mathcal{L}$ is linearly equivalent to a divisor of the form

$$N \cdot [0_C] - F$$

for some effective Cartier divisor $F \subseteq E \subseteq C$ over S (i.e., such that $F \to S$ is finite and flat). Since the "moduli space" of such divisors over S is given by the some symmetric product of copies of E (over S), it follows that (by working with universal $T = S$, $C^{\log} \to S^{\log}$, α, and F), we may assume that $D \subseteq S$ (i.e., the pull-back to S of the divisor at infinity of the moduli stack of log elliptic curves) is a *Cartier divisor* in S. But, under this assumption, one sees easily that Lemma 3.1 holds if and only if it holds over $S\backslash D$. Thus, in particular, we may assume that $D = \emptyset$.

Thus, $E \to S$ is proper and smooth. Moreover, the line bundle $\mathcal{L}(-d \cdot [0_E])$ on E is of relative degree 0, hence is *preserved (up to tensor product with a line bundle pulled back from S) by translation by α*. (This essentially amounts to the *"theorem of the square"* — cf., e.g., [Mumf5], Chapter II, §6, Corollary 4.) Writing this fact out explicitly yields the isomorphism asserted in the statement of Lemma 3.1 ◯

Thus, by *subtracting* $(\nabla')^{\otimes d}$ from the connection ∇_α on $\mathcal{L}_\alpha|_{U_\alpha}$, we obtain a connection ∇^α on $\mathcal{L}|_{U_\alpha}$, as desired. It is clear that this correspondence $(\alpha, \nabla) \mapsto \nabla^\alpha$ is *additive in α* and *functorial in T*.

Thus, we obtain a natural *action of $E^\dagger$ on $C^{\mathcal{L}}$*, as desired. Since this action is compatible with the usual action of E on C relative to the projections $E^\dagger \to E$ and $C^{\mathcal{L}} \to C$, as well as with the natural action of $W_E \subseteq E$ on the fibers of $C^{\mathcal{L}} \to C$ (where W_E is the affine group scheme over S associated to ω_E), we thus see that this action of $E^\dagger$ on $C^{\mathcal{L}}$ induces a natural structure of *$E^\dagger$-torsor on $E^{\mathcal{L}}$*. Now we make the following:

Definition 3.2. We shall refer to $E^{\mathcal{L}}$ (respectively, $C^{\mathcal{L}}$), equipped with its natural $E^\dagger$-action as defined above, as the *Hodge torsor (respectively, compactified Hodge torsor) associated to the line bundle $\mathcal{L}$*.

If $\mathcal{L}$ is defined by a Cartier divisor F on C, then we will frequently write E^F, C^F for $E^{\mathcal{L}}$, $C^{\mathcal{L}}$.

Observe that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on C of the *same relative degree over* S, then $C^{\mathcal{L}_1}$ and $C^{\mathcal{L}_2}$ *differ by a* ω_E*-torsor*

$$\mathcal{T}^{\mathcal{L}_1,\mathcal{L}_2} \to S$$

defined over S. Indeed, this follows from the fact the class in

$$\mathbf{R}^1 f_*(\omega_E|_C) \cong \mathcal{O}_S$$

defined by any $C^{\mathcal{L}}$ is simply the relative degree of $\mathcal{L}$. Thus, we will write

$$C^{\mathcal{L}_1} = \mathcal{T}^{\mathcal{L}_1,fl_2} + C^{\mathcal{L}_2}$$

(where the "+" is to be understood as the "sum of ω_E-torsors"). Similarly, since (cf. [Mzk1], Chapter III, Theorem 4.2) the class defined in $\mathbf{R}^1 f_*(\omega_E|_C) \cong \mathcal{O}_S$ by $E_C^{\dagger}$ is -1, for any line bundle $\mathcal{L}$ of relative degree -1, we shall write

$$\mathcal{T}^{\dagger,\mathcal{L}} \to S$$

for the ω_E-torsor over S given by the difference "$E_C^{\dagger} - C^{\mathcal{L}}$." When $\mathcal{L} = \mathcal{O}_C(-[0_C])$, we shall simply write

$$\mathcal{T}^{\dagger} \to S$$

for $\mathcal{T}^{\dagger,\mathcal{L}} \to S$.

In fact, the ω_E-torsor $C^{\mathcal{L}} \to C$ was discussed in [Mzk1], Chapter III, §4, in some detail, under the name "$T_{\mathrm{or}} \to C$." In the notation of the present discussion, the discussion of [Mzk1], Chapter III, §4, may be summarized as follows:

Proposition 3.3. (Relationship Between the Universal Extension and the Hodge Torsor) *The* ω_E*-torsor* $\mathcal{T}^{\dagger}$ *satisfies the following:*

(1) *The* ω_E*-torsor* $\mathcal{T}^{\dagger}$ *is naturally isomorphic to the torsor of splittings of the exact sequence*

$$0 \to \omega_E^{\otimes 2} = \mathcal{I}^2/\mathcal{I}^3 \to \mathcal{I}/\mathcal{I}^3 \to \omega_E = \mathcal{I}/\mathcal{I}^2 \to 0$$

(where we write $\mathcal{I}$ for the sheaf of ideals on C that defines the zero section). Moreover, the torsor $2 \cdot \mathcal{T}^\dagger$ (obtained from $\mathcal{T}^\dagger$ by pushing out via the map $[2] : \omega_E \to \omega_E$ given by multiplication by 2) admits a **canonical trivialization**. *The splitting of the above exact sequence that corresponds to this trivialization may be characterized as either (i) the unique splitting that is compatible with the natural action of ± 1 on the above exact sequence (cf. [Mzk1], Chapter III, §4); or (ii) the splitting defined by integrating invariant differentials on E (cf. [Mzk1], Chapter IX, Appendix).*

(2) *There is a canonical isomorphism of ω_E-torsors over C:*

$$E^\dagger_C \cong C^{-[O_C]} + \mathcal{T}^\dagger$$

(which is valid **even if 2 is not invertible on** *S). Thus, if 2 is invertible on S, we obtain an isomorphism $E^\dagger_C \cong C^{-[O_C]}$ (cf. [Mzk1], Chapter III, Theorem 4.2).*

Finally, (as was observed in [Mzk1], Chapter III, the Remark following Corollary 3.3) in general, the torsor $\mathcal{T}^\dagger$ is **nontrivial**.

Before concluding this §, we also note the following:

Proposition 3.4. **(Relationship Between Different Hodge Torsors)** *Let $\alpha \in E(S)$. Write $\mathcal{L}_\alpha$ for the line bundle on C obtained by translating $\mathcal{L}$ by α. Then "transport of structure" induces natural isomorphisms*

$$C^{\mathcal{L}} \cong C^{\mathcal{L}_\alpha}; \quad E^{\mathcal{L}} \cong E^{\mathcal{L}_\alpha}$$

compatible with the actions of $E^\dagger$ on both sides, and (via the projections to C on both sides) with translation by α on C.

If, moreover, α is a **torsion point of order** *m, and $m \in \mathcal{O}_S^\times$, then then there exists a unique torsion point $\alpha^\dagger \in E^\dagger(S)$ lifting α. Moreover, transport of structure via translation by $-\alpha$, followed by the action of $\alpha^\dagger \in E^\dagger(S)$ on $C^{\mathcal{L}}$, defines a natural* **isomorphism**

$$C^{\mathcal{L}_\alpha} \cong C^{\mathcal{L}}$$

compatible with the projections on both sides to C (as well as with the actions on both sides of $E^\dagger$). In particular, in this case, the torsor $\mathcal{T}^{\mathcal{L},\mathcal{L}_\alpha} \to S$ admits a natural splitting.

Proof. There is practically nothing here that requires further argument. Note that the existence of a lifting $\alpha^\dagger$ as asserted follows from the fact that the kernel W_E of $E^\dagger \to E$ is the affine group scheme associated to the line bundle ω_E on S, together with the assumption that $m \in \mathcal{O}_S^\times$. $\bigcirc$

Remark. Suppose that $\alpha \in E(S)$ is a *torsion point of order* m, and that $2m \in \mathcal{O}_S$ is a *non-zero divisor*. Then, by *identifying* the restrictions to $S \otimes \mathbf{Z}[\frac{1}{m}]$ of $C^{-[0_C]}$ and $C^{-[\alpha]}$ via the isomorphism at the end of Proposition 3.4 involving $\alpha^\dagger$, and using the natural splitting of $\mathcal{T}^\dagger$ discussed in Proposition 3.3, we see that

> *One may think of* $C^{-[\alpha]} \cong C^\alpha$ *as a sort of "modified integral structure" on* $C^{-[0_C]} \otimes \mathbf{Z}[\frac{1}{m}]$ *or, alternatively, on* $E_C^\dagger \otimes \mathbf{Z}[\frac{1}{2m}]$.

Of course, all of these integral structures are compatible with the natural actions of $E^\dagger$, and the natural projections to C.

Remark. In fact, just as we defined *étale integral structures* on $E^\dagger$, $E_C^\dagger$ in §1, we would also like to define étale integral structures on the (compactified) Hodge torsors of the present §, in such a way that the various actions of $E^\dagger$ are compatible with the various étale integral structures. This is precisely the topic of §4, below.

§4. The Étale Integral Structure on the Hodge Torsors

In this §, we show that the *compactified Hodge torsors* introduced in §3 admit natural *étale integral structures*, similar to those constructed on the universal extension in §1. In fact, in the present §, we will only give the proof of the existence of such integral structures for degenerating elliptic curves and elliptic curves with *ordinary reduction* modulo p. In order to complete the proof for arbitrary elliptic curves (which we will do in §8), we will need to enlist the aid of the techniques of §6, 7, below. (Note: The results of §6, 7, that will be necessary to complete the proof will only require the étale integral structure for degenerating elliptic curves, so there will not be any "vicious circles" in the reasoning.)

§4.1. Notation and Set-Up:

In this §, we use notation similar to that of [Mzk1], Chapter IV, §2: More precisely, let $\mathcal{O}$ be a Zariski localization of the ring of integers of a finite extension of $\mathbf{Q}$; let m be a *positive integer*; $n \stackrel{\text{def}}{=} 2m$;

$$A \stackrel{\text{def}}{=} \mathcal{O}[[q^{\frac{1}{n}}]]; \quad S \stackrel{\text{def}}{=} \mathrm{Spec}(A); \quad \widehat{S} \stackrel{\text{def}}{=} \mathrm{Spf}(A)$$

(where q is an indeterminate, and we regard A as equipped with the q-adic topology). Over S, we have a *natural degenerating elliptic curve* $E \to S$ (with "Tate parameter" given by

q), together with a *compactification* $C \to S$ whose pull-back $C_{\widehat{S}} \to \widehat{S}$ to $\widehat{S}$ may be obtained as the *quotient of an object* $C^{\infty}_{\widehat{S}}$ *with respect to the natural action of a group*, which we denote by $\mathbf{Z}_{\mathrm{et}} \stackrel{\mathrm{def}}{=} \mathbf{Z}$, on $C^{\infty}_{\widehat{S}}$:

$$C_{\widehat{S}} = C^{\infty}_{\widehat{S}}/\mathbf{Z}_{\mathrm{et}}$$

Here, $C^{\infty}_{\widehat{S}}$ is the pull-back to $\widehat{S}$ of the *"Néron model"* of $(\mathbf{G}_{\mathrm{m}})_{\mathcal{O}[[q]][q^{-1}]}$ over $\mathcal{O}[[q]]$. Thus, the *special fiber* $(C^{\infty}_{\widehat{S}})_{\mathrm{spl}}$ of $C^{\infty}_{\widehat{S}}$ (i.e., the zero locus of the function $q^{\frac{1}{n}}$) is a *chain of* $\mathbf{P}^1$*'s* over $\mathrm{Spec}(\mathcal{O})$ indexed by $\mathbf{Z}$ (which we think of as the group of exponents of q that occur in $(\mathcal{O}[[q]][q^{-1}])^{\times}$) and permuted by the action of the group $\mathbf{Z}_{\mathrm{et}}$, in a fashion which is compatible with the indexing by $\mathbf{Z}$ and the natural action of $\mathbf{Z}_{\mathrm{et}} = \mathbf{Z}$ on $\mathbf{Z}$ (by addition). Note, moreover, that we have a natural identification $E_{\widehat{S}} = (\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$. We will denote the usual *multiplicative coordinate* on this copy of $\mathbf{G}_{\mathrm{m}}$ by U.

In the following discussion, in addition to $C \to S$, we will also need to make use of the log elliptic curve $\widetilde{C} \to S$ whose Tate parameter is given by $q^{\frac{1}{n}}$. Let us denote by $\widetilde{E}$, $\widetilde{C}^{\infty}_{\widehat{S}}$, $\widetilde{U}$, etc. the various objects (analogous to E, $C^{\infty}_{\widehat{S}}$, U, etc.) associated to $\widetilde{C}$. Also, let us observe that we have a natural *isogeny* of degree n:

$$\widetilde{C} \to C$$

defined by $U \mapsto \widetilde{U}^n$. Thus, this isogeny is covered by an isogeny $\widetilde{C}^{\infty}_{\widehat{S}} \to C^{\infty}_{\widehat{S}}$ (also of degree n, and defined by $U \mapsto \widetilde{U}^n$). Put another way, instead of introducing $\widetilde{C}$, etc., we could simply have said that occasionally we will also make use of the n-th root $U^{\frac{1}{n}}$ of U.

Next, we consider *line bundles*. Write

$$\mathcal{L}_C \stackrel{\mathrm{def}}{=} \mathcal{O}_C(0_C); \quad \mathcal{L}_{\widetilde{C}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{\widetilde{C}}([(\boldsymbol{\mu}_n)_{\widetilde{C}}])$$

(where 0_C is the zero section of $C \to S$, and $(\boldsymbol{\mu}_n)_{\widetilde{C}} \subseteq \widetilde{C}$ is the kernel of the isogeny $\widetilde{E} \to E$). Thus, the pull-back of $\mathcal{L}_C$ to $\widetilde{C}$ is given by $\mathcal{L}_{\widetilde{C}}$.

Now it follows from the theory of [Mzk1], Chapter IV, §2, that $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}} \stackrel{\mathrm{def}}{=} \mathcal{L}|_{\widetilde{C}^{\infty}_{\widehat{S}}}$ *admits a natural trivialization* "θ^m." Moreover, the line bundle $\mathcal{L}|_{\widetilde{C}^{\infty}_{\widehat{S}}}$ admits a *natural action* of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ (which is compatible with the natural action of $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ on $\widetilde{C}^{\infty}_{\widehat{S}}$). Relative to this action, $\boldsymbol{\mu}_n$ acts trivially on θ^m, while $k_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ maps $\theta^m \mapsto q^{\frac{1}{2}k^2} \cdot U^k \cdot \theta^m$ (cf. [Mzk1], Chapter IV, Proposition 2.2). Moreover, $\mathcal{L}_C$ is obtained from $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$ by descending $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$ by means of the action obtained by tensoring this natural action with a certain *character* $\chi_{\theta} : \mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$ (cf. [Mzk1], Chapter IV, Theorem 2.1) of order two.

In the present discussion, we would like to assume further (cf. [Mzk1], Chapter V, §4) that we are given a *character*

$$\chi_{\mathcal{L}} \in \mathrm{Hom}(\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$$

We would then like to consider (cf. [Mzk1], Chapter V, §4) what is, in effect, the result of *twisting* $\mathcal{L}_C$ *by* $\chi_{\mathcal{L}}$. In more down to earth terms, this amounts to considering sections of $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$ which are invariant which respect to the action of (various subgroups of) $\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n$ given by tensoring the natural action with the character $\chi_{\mathcal{M}} \stackrel{\mathrm{def}}{=} \chi_{\mathcal{L}} \cdot \chi_{\theta}$. (Here, the "subgroup" is the subgroup corresponding to the intermediate covering of $\widetilde{C}^{\infty}_{\widehat{S}} \to C_{\widehat{S}}$ over which the section is to be defined.) In the following discussion, for the sake of brevity:

> *We shall denote by "$\mathcal{L}^{\chi}_{C}$" the object consisting of $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$, equipped with the action of $\chi_{\mathcal{M}}$.*

In particular, we shall refer to $\chi_{\mathcal{M}}$-invariant sections of $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$ as "sections of $\mathcal{L}^{\chi}_{C}$." Observe that, by applying finite flat descent to $\mathcal{L}_{\widetilde{C}^{\infty}_{\widehat{S}}}$ (equipped with the action defined by $\chi_{\mathcal{M}}$) and then algebrizing, over $E \subseteq C$, as well as over $\widetilde{C}$, we obtain *genuine line bundles* "$\mathcal{L}^{\chi}_{E}$," "$\mathcal{L}^{\chi}_{\widetilde{C}}$." (Note that we cannot apply such a descent argument to $\widetilde{C} \to C$ since it *fails to be flat at the nodes.*)

Thus, the space

$$\Gamma(C^{\infty}_{\widehat{S}}, \mathcal{L}^{\chi}_{C^{\infty}_{\widehat{S}}})$$

may be identified with the *topological A-linear combinations of*

$$U^k \cdot \widetilde{U}^{i_\chi} \cdot \theta^m$$

where $i_\chi \in \{-m, -m+1, \ldots, m-1\}$ is a number that depends only on the character $\chi_{\mathcal{L}}$ (cf. [Mzk1], Chapter V, Theorem 4.8), and k ranges over all integers. Moreover, for $k \in \mathbf{Z}$, the corresponding element $k_{\mathrm{et}} \in \mathbf{Z}_{\mathrm{et}}$ acts on these sections by:

$$\theta^m \mapsto \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{k^2}{2}} \cdot U^k \cdot \theta^m; \quad \widetilde{U} \mapsto q^{\frac{k}{n}} \cdot \widetilde{U}; \quad U \mapsto q^k \cdot U;$$

Thus, a generator of $\Gamma(C, \mathcal{L}_C)$ is given by the series

$$\sum_{k \in \mathbf{Z}} \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{1}{2}k^2 + \frac{i_\chi}{n}k} \cdot U^k \cdot \widetilde{U}^{i_\chi} \cdot \theta^m$$

(cf. [Mzk1], Chapter V, Theorem 4.8).

§4.2. Degenerating Elliptic Curves:

Just as $E^\dagger$ and $E^\dagger_C$ admit natural *étale integral structures* (cf. the discussion of §1), the compactified Hodge torsors $C^{\mathcal{L}}$ introduced in §3 also admit natural *étale integral structures.* Also, just as in the case of $E^\dagger_C$, we will construct these integral structures by first constructing them in a *formal neighborhood of infinity* (cf. the discussion below) and then extending them to the rest of the moduli stack of elliptic curves.

First, let us observe (cf. the discussion preceding Lemma 1.1) that by using the splitting $\kappa_{C^\infty_{\widehat{S}}} : C^\infty_{\widehat{S}} \to E^\dagger_{C^\infty_{\widehat{S}}} \stackrel{\text{def}}{=} E^\dagger_C|_{C^\infty_{\widehat{S}}}$ (defined by the canonical section "κ" of [Mzk1], Chapter III, Theorem 2.1) and the logarithmic differential $d\log(U)$, one may think of the push-forward $\mathcal{R}$ of the structure sheaf $\mathcal{O}_{E^\dagger_{C^\infty_{\widehat{S}}}}$ to $C^\infty_{\widehat{S}}$ as being given by a *polynomial algebra*:

$$\mathcal{R} = \mathcal{O}_{C^\infty_{\widehat{S}}}[T]$$

(where the indeterminate T is that defined by the trivialization of ω_E given by $d\log(U)$). Then we would like to define the $\chi_{\mathcal{L}}$-*étale integral structure on* $\mathcal{R}$ by:

$$\mathcal{R}^{\text{et}}_\chi \stackrel{\text{def}}{=} \bigoplus_{r \geq 0} \mathcal{O}_{C^\infty_{\widehat{S}}} \cdot T^{[r]}_\chi$$

where

$$T^{[r]}_\chi \stackrel{\text{def}}{=} \binom{T - (i_\chi/n)}{r}$$

(cf. the integral structure

$$\mathcal{R}^{\text{et}} \stackrel{\text{def}}{=} \bigoplus_{r \geq 0} \mathcal{O}_{C^\infty_{\widehat{S}}} \cdot T^{[r]}$$

defined by the étale integral structure on $E^\dagger_{C,\text{et}}$).

Observe (by applying Lemma 1.1, "shifted over by i_χ/n") that $\mathcal{R}^{\text{et}}_\chi$ is closed under multiplication, hence defines an $\mathcal{O}_{C^\infty_{\widehat{S}}}$-*algebra.* Moreover, the corresponding geometric object $Spec(\mathcal{R}^{\text{et}}_\chi)$ over $C^\infty_{\widehat{S}}$ forms a *torsor* over the $C^\infty_{\widehat{S}}$-group (cf. Lemma 1.1, (ii)) object $Spec(\mathcal{R}^{\text{et}})$. In more concrete terms:

This torsor amounts to the $\mathbf{Z}$*-torsor* $(i_\chi/n) + \mathbf{Z}$.

That is, $\mathcal{R}^{\mathrm{et}}_\chi$ (respectively, $\mathcal{R}^{\mathrm{et}}$) may be thought of as the ring of $\mathcal{O}_{C^\infty_{\widehat{S}}}$-valued functions on $(i_\chi/n) + \mathbf{Z}$ (respectively, $\mathbf{Z}$).

Note that over the "generic fiber" $E_U \stackrel{\mathrm{def}}{=} E \otimes_A A[q^{-1}]$, the notation $\mathcal{L}^\chi_{E_U}$ corresponds to a genuine line bundle of the form $\mathcal{O}_{E_U}(\eta)$, for some torsion point $\eta \in E(A[q^{-1}])$ annihilated by n. Then let us observe that, *over* $A[q^{-1}]$, the integral structure on $E^\dagger$ defined by

$$\bigoplus_{r \geq 0} \mathcal{O}_{C^\infty_{\widehat{S}}} \cdot \{T - (i_\chi/n)\}^r$$

coincides with the integral structure on $E^\dagger$ *defined by* $C^{[\eta]}$ (cf. the first Remark following Proposition 3.4). Indeed, to see this, we argue as follows: First, note that the relative connection (i.e., over $\widehat{S}$) $\nabla_{C^\infty_{\widehat{S}}}$ on $\mathcal{L}^\chi_{\widetilde{C}^\infty_{\widehat{S}}} = \mathcal{L}_{\widetilde{C}^\infty_{\widehat{S}}}$ obtained by pull-back (via the canonical section) from the *tautological (cf. [Mzk1], Chapter III, Theorem 4.2) relative connection* $\nabla_{E^\dagger_{C^\infty_{\widehat{S}}}}$ on $E^\dagger_{C^\infty_{\widehat{S}}}$ is that for which θ^m is horizontal (cf. [Mzk1], Chapter III, Theorem 5.6). But this connection becomes integral on $\Gamma(C^\infty_{\widehat{S}}, \mathcal{L}^\chi_{C^\infty_{\widehat{S}}})$ (i.e., on sections of the form $U^k \cdot \widetilde{U}^{i_\chi} \cdot \theta^m$) if and only if it is supplemented by the extra term "$-(i_\chi/n)$." On the other hand, since $\nabla_{E^\dagger_{C^\infty_{\widehat{S}}}}$ may be written as $\nabla_{C^\infty_{\widehat{S}}} + T$ (cf. the discussion of [Mzk1], Chapter III, §7), it thus follows that the integral structure on $E^\dagger$ defined by $C^{[\eta]}$ is that given by *polynomials in* $T - (i_\chi/n)$, as desired (cf. also [Mzk1], Chapter IV, Theorem 3.3).

Note that the $\mathcal{R}^{\mathrm{et}}_\chi$ is *preserved* by the natural action on $\mathcal{R}$ of $\mathbf{Z}_{\mathrm{et}}$ (for which $1_{\mathrm{et}}(T) = T + 1$). Thus, $\mathcal{R}^{\mathrm{et}}_\chi$ descends to an integral structure on $E^\dagger_C$, which we denote by

$$C^{[\eta]}_{\mathrm{et}}$$

i.e., $C^{[\eta]}_{\mathrm{et}}|_{C^\infty_{\widehat{S}}} = \mathit{Spec}(\mathcal{R}^{\mathrm{et}}_\chi)$ (where the "Spec" is as an object over $C^\infty_{\widehat{S}}$). In particular, it is natural to regard $C^{[\eta]}_{\mathrm{et}}$ as the *analogue for* $C^{[\eta]}$ *of the étale integral structure* $E^\dagger_{C,\mathrm{et}}$ *on* $E^\dagger_C$.

Finally, before proceeding, we note that the integral structure $\mathcal{R}^{\mathrm{et}}_\chi$ also satisfies the following properties:

(1) *The integral structure* $\mathcal{R}^{\mathrm{et}}_\chi$ *is completely determined by the torsion point* η*.* In particular, it is independent of the choice of n, $\chi_{\mathcal{L}}$ (i.e., among those n, $\chi_{\mathcal{L}}$ that give rise to the same twist of the line bundle $\mathcal{L}_C$ as η).

(2) *The sections* $\zeta_0^{\mathrm{CG}}, \zeta_1^{\mathrm{CG}}, \ldots, \zeta_r^{\mathrm{CG}}, \ldots \in \Gamma((E^{\dagger}_{C,\mathrm{et}})_{\widehat{S}}, \mathcal{L}^{\chi}_{(E^{\dagger}_{C,\mathrm{et}})_{\widehat{S}}}) \otimes \mathbf{Q}$ *of [Mzk1], Chapter V, Theorem 4.8, form a basis over* A *of the module* $\Gamma(C^{[\eta]}_{\mathrm{et}}, \mathcal{L}^{\chi}|_{C^{[\eta]}_{\mathrm{et}}})$. Indeed, this follows from the description of ζ_r^{CG} given in [Mzk1], Chapter V, Theorem 4.8, as the result of applying the operator

$$\binom{\nabla_{C^{\infty}_{\widehat{S}}} + \lambda_r - (i_{\chi}/n)}{r}$$

(cf. the definition of $T^{[r]}_{\chi}$ above and the fact that "λ_r" is an *integer*) to ζ_0^{CG}, together with [Mzk1], Chapter III, Theorem 5.6.

§4.3. Ordinary Elliptic Curves:

In the following, we maintain the notation of the discussion of the Verschiebung morphism in §2. Write $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(0_E)$ for the line bundle defined by the origin of E. Since the pull-back of $\mathcal{L}$ by $\mathcal{V}^n : E^{\mathrm{F}^n} \to E$ has degree p^n, the resulting *theta group* (cf. [Mumf1,2,3]; [Mumf5], §23; or, alternatively, [Mzk1], Chapter IV, §1, for an exposition of the theory of theta groups) fits into an exact sequence:

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}_{(\mathcal{V}^n)^*\mathcal{L}} \to (E^{\mathrm{F}^n})[p^n] \to 0$$

of group schemes over S. Next, observe that since $\mathcal{V}^n$ is *étale*, the natural morphism

$$(E^{\mathrm{F}^n})[p^n]^{\boldsymbol{\mu}} \to E[p^n]^{\boldsymbol{\mu}}$$

is an *isomorphism*. Thus, we get a natural inclusion

$$E[p^n]^{\boldsymbol{\mu}} \hookrightarrow (E^{\mathrm{F}^n})[p^n] \subseteq E^{\mathrm{F}^n}$$

If we pull-back the above exact sequence via this inclusion, we thus obtain an exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}_n \to E[p^n]^{\boldsymbol{\mu}} \to 0$$

of group schemes over S. Note, moreover, that since the commutator of $\mathcal{G}_{(\mathcal{V}^n)^*\mathcal{L}}$ clearly vanishes on the "cyclic" group scheme $E[p^n]^{\boldsymbol{\mu}}$, it follows that $\mathcal{G}_n$ is *abelian*. Also, let us observe that ± 1 acts naturally on all the objects (i.e., E^{F^n}, $\mathcal{L}$, etc.) involved in the definition of $\mathcal{G}_n$. We thus obtain a natural action of ± 1 on $\mathcal{G}_n$ itself which is compatible with the above exact sequence, and induces the *trivial action* (respectively, the *usual action of* ± 1) on $\mathbf{G}_{\mathrm{m}}$ (respectively, $E[p^n]^{\boldsymbol{\mu}}$).

Now let us write $E_{\boldsymbol{\mu}}$ for the inductive limit of the $E[p^n]^{\boldsymbol{\mu}}$. Then it is clear that the formation of $\mathcal{G}_n$ is compatible with the morphisms of the "Verschiebung tower" of E (cf. §2), hence that we obtain natural inclusions $\mathcal{G}_n \hookrightarrow \mathcal{G}_{n+1}$, which induce the identity on $\mathbf{G}_{\mathrm{m}}$ and the natural inclusion $E[p^n]^{\boldsymbol{\mu}} \hookrightarrow E[p^{n+1}]^{\boldsymbol{\mu}}$ on $E[p^n]^{\boldsymbol{\mu}}$. Thus, passing to the limit, we obtain an exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}_\infty \to E_{\boldsymbol{\mu}} \to 0$$

of group objects over S (equipped with an action by ± 1). If we then take the "Cartier dual" of this exact sequence (i.e., apply the functor $\mathrm{Hom}(-, \mathbf{G}_{\mathrm{m}})$), we obtain an exact sequence

$$0 \to E_{\mathrm{et}} \to M_\infty \to \mathbf{Z} \to 0$$

of étale local systems on S equipped with an action by ± 1. In particular, if we push-forward this exact sequence by the morphism $[2] : E_{\mathrm{et}} \to E'_{\mathrm{et}} \stackrel{\mathrm{def}}{=} E_{\mathrm{et}}$ (i.e., multiplication by 2 on E_{et}), the resulting exact sequence

$$0 \to E'_{\mathrm{et}} \to M'_\infty \to \mathbf{Z} \to 0$$

admits a unique splitting compatible with the action by ± 1 (which acts trivially on $\mathbf{Z}$, and as ± 1 on E'_{et}). Returning to the category of group objects, we thus obtain that if we pull-back $\mathcal{G}_\infty$ by the morphism $[2] : E'_{\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} E_{\boldsymbol{\mu}} \to E_{\boldsymbol{\mu}}$ given by multiplication by 2 on $E_{\boldsymbol{\mu}}$, we obtain an *exact sequence of group objects*

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}'_\infty \to E'_{\boldsymbol{\mu}} \to 0$$

which admits a unique splitting compatible with the action by ± 1. Note that this splitting may be interpreted as a *symmetric trivialization* (where by "symmetric," we mean compatible with the action by ± 1)

$$t'_{\mathcal{L}} : (\mathcal{L}|_{0_E})|_{E'_{\boldsymbol{\mu}}} \cong \mathcal{L}|_{E'_{\boldsymbol{\mu}}}$$

which restricts to the identity over 0_E. Also, we observe that:

(1) Since $t'_{\mathcal{L}}$ is uniquely characterized by the fact that it is symmetric, it follows that in a neighborhood of infinity, $t'_{\mathcal{L}}$ *coincides with the trivialization by the section* "θ" (cf. the discussion of §4.1, 4.2, in the case $n = 2$; $\chi_{\mathcal{L}}$ trivial), since this trivialization is also symmetric.

(2) In general, when $p = 2$, the trivialization $t'_{\mathcal{L}}$ *does not descend to* $E_{\boldsymbol{\mu}}$. Indeed, by (1) above, this assertion may be checked in a neighborhood

of infinity, where it amounts to the fact that the multiplicative portion of the character "χ" of [Mzk1], Chapter IV, Theorem 2.1, is *nontrivial.*

Thus, in particular, (2) implies that the restriction of the unique symmetric splitting of $\mathcal{G}'_\infty \to E'_{\boldsymbol{\mu}}$ to the kernel $\boldsymbol{\mu}'_2 \subseteq E'_{\boldsymbol{\mu}}$ of $E'_{\boldsymbol{\mu}} \to E_{\boldsymbol{\mu}}$ differs from the homomorphism $\boldsymbol{\mu}'_2 \to \mathcal{G}_\infty$ defined by the fact that $\mathcal{L}|_{E'_{\boldsymbol{\mu}}}$ is obtained via pull-back from $E_{\boldsymbol{\mu}}$ by the unique nontrivial character $\boldsymbol{\mu}'_2 \to \mathbf{G}_\mathrm{m} \subseteq \mathcal{G}'_\infty$.

Next, observe that since $E[p]^\mathrm{et}/(2)$ (i.e., the quotient of $E[p]^\mathrm{et}$ by $2 \cdot E[p]^\mathrm{et}$) is an étale local system on S of cyclic groups (i.e., the trivial group if p is odd; the group $\mathbf{Z}/2\mathbf{Z}$ if $p = 2$), it follows that $E[p]^\mathrm{et}/(2)$ admits a *unique generating section* over S. Let us write

$$\widehat{T} \to S$$

for the p-adic completion of the profinite étale covering of S given by the E_et-torsor $T \to S$ of liftings of the unique generating section of $E[p]^\mathrm{et}/(2)$ relative to the natural surjection $E'_\mathrm{et} \cong E_\mathrm{et} \to E[p]^\mathrm{et}/(2)$. Observe that this torsor $T \to S$ also admits the following interpretation: If p is *odd*, then let $\mathcal{M}$ be the trivial line bundle on $E_{\boldsymbol{\mu}}$. If $p = 2$, then the morphism

$$[2] : E'_{\boldsymbol{\mu}} \to E_{\boldsymbol{\mu}}$$

may be regarded as a $\boldsymbol{\mu}_2$-torsor over $E_{\boldsymbol{\mu}}$. In particular, this $\boldsymbol{\mu}_2$-torsor corresponds to a line bundle $\mathcal{M}$ on $E_{\boldsymbol{\mu}}$ equipped with an isomorphism $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_{E_{\boldsymbol{\mu}}}$. Then, *regardless of the parity of* p, $T \to S$ may be interpreted as the $\mathrm{Hom}(E_{\boldsymbol{\mu}}, \mathbf{G}_\mathrm{m}) = E_\mathrm{et}$-torsor of trivializations of $\mathcal{M}$ by a section of $\mathcal{M}$ over $E_{\boldsymbol{\mu}}$ that corresponds to a character $\in \mathrm{Hom}(E'_{\boldsymbol{\mu}}, \mathbf{G}_\mathrm{m}) = E'_\mathrm{et}$ of $E'_{\boldsymbol{\mu}}$.

In particular, we see that — *regardless of the parity of* p — we have a *tautological trivializing section*

$$s_\mathcal{M} \in \mathcal{M}|_{\widehat{T}}$$

of $\mathcal{M}$ over $\widehat{T}$.

Now write

$$\mathcal{P} \stackrel{\mathrm{def}}{=} \mathcal{L} \otimes_{\mathcal{O}_S} (\mathcal{L}|_{0_E}^{-1})$$

Thus, since $t'_\mathcal{L}$ defines a section of $\mathcal{P}|_{E'_{\boldsymbol{\mu}}}$ on which $\boldsymbol{\mu}'_2$ acts (cf. the discussion above) via the unique nontrivial character, we thus obtain that $t'_\mathcal{L}$ may be regarded as a *section of* $\mathcal{P} \otimes \mathcal{M}$ *over* $E_{\boldsymbol{\mu}}$. In particular, by applying the trivialization $s_\mathcal{M}$ of $\mathcal{M}$ over $\widehat{T}$, we see that $t'_\mathcal{L}$ defines a trivialization

$$t_{\mathcal{L}} : (\mathcal{L}|_{0_E})|_{E_{\boldsymbol{\mu}}|_{\widehat{T}}} \cong \mathcal{L}|_{E_{\boldsymbol{\mu}}|_{\widehat{T}}}$$

over $\widehat{T}$.

Next, let us write $\mathcal{I}$ for the coherent sheaf of ideals on E such that $V(\mathcal{I}) = 0_E$, and

$$E_\epsilon \stackrel{\text{def}}{=} V(\mathcal{I}^2) \subseteq E$$

for the first nontrivial infinitesimal neighborhood of 0_E in E. Note that the difference between the tautological point $\in E(E_\epsilon \otimes \mathbf{Z}/p^n\mathbf{Z})$ and the origin 0_E is *annihilated by* p^n, hence lies inside $E[p^n]^{\boldsymbol{\mu}}$. Thus, we obtain that $E_\epsilon \otimes \mathbf{Z}/p^n\mathbf{Z} \subseteq E[p^n]^{\boldsymbol{\mu}}$, hence (by passing to the p-adic limit) we have

$$E_\epsilon \subseteq E^{\wedge}_{\boldsymbol{\mu}}$$

(where $E^{\wedge}_{\boldsymbol{\mu}}$ denotes the p-adic completion of $E_{\boldsymbol{\mu}}$). In particular, if we restrict (the p-adic completion of) $t_{\mathcal{L}}$ to E_ϵ, we obtain a trivialization

$$t_\epsilon : (\mathcal{L}|_{0_E})|_{(E_\epsilon)_{\widehat{T}}} \cong \mathcal{L}|_{(E_\epsilon)_{\widehat{T}}}$$

But observe that *such a trivialization is none other than a connection (relative to the morphism $E \to S$) on the line bundle $\mathcal{L}$ at the point* 0_E. Thus, by the definition of the Hodge torsor $E^{\mathcal{L}}$ (cf. §3), we obtain that t_ϵ defines a *natural symmetric section*

$$s^{\text{ord}} : \widehat{T} \to E^{\mathcal{L}}$$

of the Hodge torsor $E^{\mathcal{L}}$ over $\widehat{T}$.

Now since $E^\dagger$ acts on $E^{\mathcal{L}}$, the section s^{ord} thus determines a *natural symmetric isomorphism*

$$E^{\mathcal{L}}|_{\widehat{T}} \cong E^\dagger|_{\widehat{T}}$$

Moreover, we have the following result:

Lemma 4.1. *If we equip $E^{\mathcal{L}}|_{\widehat{T}}$ with the integral structure obtained by transporting the étale integral structure on $E^\dagger|_{\widehat{T}}$ via the above isomorphism, then, in a neighborhood of infinity, this integral structure coincides with the étale integral structure defined on $E^{\mathcal{L}}$ in the discussion of §4.2 (for trivial $\chi_{\mathcal{L}}$). In particular, this integral structure descends from $\widehat{T}$ to S. Finally, if p is* **odd**, *then this integral structure coincides with the integral* *structure obtained by transporting the étale integral structure on $E^{\dagger}_{\mathrm{et}}$ via the isomorphism $E^{\dagger} \cong E^{\mathcal{L}}$ discussed at the end of Proposition 3.3, (2).*

Proof. Indeed, in a neighborhood of infinity, $t'_{\mathcal{L}}$ corresponds to the trivialization θ. Moreover, the torsor $T \to S$ corresponds to (the result of pushing out via $\mathbf{Z} \to \mathbf{Z}_p$) the $\mathbf{Z}$-torsor of liftings $a \in \frac{1}{2} \cdot \mathbf{Z}$ of the unique nonzero element of $\frac{1}{2} \cdot \mathbf{Z}/\mathbf{Z}$. The trivialization $t_{\mathcal{L}}$ obtained by "dividing" $t'_{\mathcal{L}}$ by $s_{\mathcal{M}}$ then corresponds to the trivialization in a neighborhood of infinity given by $U^{-a} \cdot \theta$. Since the coordinate "T" of the discussion of §5 corresponds to the splitting of $E^{\mathcal{L}}_{\mathbf{Q}}$ defined by the connection for which θ is horizontal, it thus follows that "$T - a$" corresponds to the splitting of $E^{\mathcal{L}}_{\mathbf{Q}}$ defined by the connection for which $U^{-a} \cdot \theta$ is horizontal. In particular, we see that the integral structure on $E^{\mathcal{L}}|_{\widehat{T}}$ obtained by transporting the étale integral structure on $E^{\dagger}|_{\widehat{T}}$ via the isomorphism $E^{\mathcal{L}}|_{\widehat{T}} \cong E^{\dagger}|_{\widehat{T}}$ amounts to that defined by the "$\binom{T-a}{r}$." But this integral structure is (by elementary properties of the binomial coefficient polynomials) the same as that defined by the "$\binom{T-a'}{r}$" for any $a \in \frac{1}{2} \cdot \mathbf{Z}_p$ that lifts the unique nonzero element of $\frac{1}{2} \cdot \mathbf{Z}_p/(2)$. In particular, this integral structure is the same as that defined by the "$\binom{T+\frac{1}{2}}{r}$." Since, in the present situation, $\chi_{\mathcal{L}}$ is trivial, it thus follows that (in the notation of §5) $i_{\chi}/n = -\frac{1}{2}$, hence that the two integral structures in question coincide. Since this integral structure is independent of the choice of lifting a, we thus see that this integral structure also descends from $\widehat{T}$ to S, as desired. Finally, if p is *odd*, then the fact that the integral structure under consideration coincides with the integral structure obtained by transporting the étale integral structure on $E^{\dagger}_{\mathrm{et}}$ via the isomorphism $E^{\dagger} \cong E^{\mathcal{L}}$ discussed at the end of Proposition 3.3, (2), follows from the fact that, if p is odd, then the integral structure defined by the "$\binom{T+\frac{1}{2}}{r}$" is the same as the integral structure defined by the "$\binom{T}{r}$." ○

Remark. It would be nice if a construction of t_{ϵ} could be found that works without the hypothesis of ordinariness. Such a construction would alleviate the need for treating the ordinary and non-ordinary cases separately, as we have done here. Unfortunately, it appears unlikely that such a construction is possible.

§4.4. The General Case:

In the following discussion, we let S be étale over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$, and $E \to S$ be the pull-back from $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ of the *tautological elliptic curve* over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$. Also, let us write $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(0_E)$, and $S^{\mathrm{ord}} \subseteq S$ for the open subscheme obtained by *removing the supersingular points* in characteristic p. In particular, we do *not* assume that the reduction modulo p of $E \to S$ is a family of ordinary elliptic curves. In the following discussion, we would like to show that *the étale integral structure on $E^{\mathcal{L}}$ constructed in Lemma 4.1 over $\widehat{S}^{\mathrm{ord}}$ extends (necessarily uniquely) over S*. (Here and in the following, "$\wedge$" will be used to denote p-adic completion.)

Write $\mathcal{M}$ for a line bundle of the form $\mathcal{L}^{\otimes N}$ (for a positive integer N). Since $\mathcal{L}|_{E^{\mathcal{L}}}$ is *ample* on $E^{\mathcal{L}}$, working with integral structures on $E^{\mathcal{L}}$ is equivalent to working with (compatible) integral structures on

$$\mathcal{H}_{\mathcal{M}} \stackrel{\text{def}}{=} f_*(\mathcal{M}|_{E^{\mathcal{L}}})$$

(for various N). Note that $\mathcal{H}_{\mathcal{M}}$ has a natural filtration $F^r(\mathcal{H}_{\mathcal{M}}) \subseteq \mathcal{H}_{\mathcal{M}}$, consisting of those sections of torsorial degree $< r$. Thus, we have a natural isomorphism

$$(F^{r+1}/F^r)(\mathcal{H}_{\mathcal{M}}) \cong \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{M})$$

In particular, $F^r(\mathcal{H}_{\mathcal{M}})$ is a vector bundle on S of rank r.

Next, let us observe that, since S is a *regular scheme of dimension* 2, it follows that the étale integral structure on $E^{\mathcal{L}}|_{\widehat{S}^{\text{ord}}}$ constructed in Lemma 4.1 (which gives rise to an integral structure on $\mathcal{H}_{\mathcal{M}}|_{\widehat{S}^{\text{ord}}}, F^r(\mathcal{H}_{\mathcal{M}})|_{\widehat{S}^{\text{ord}}}$) already defines a (unique!) integral structure on $\mathcal{H}_{\mathcal{M}}$, $F^r(\mathcal{H}_{\mathcal{M}})$. Thus, we obtain quasi-coherent sheaves $\mathcal{H}^{\text{et}}_{\mathcal{M}}$, $F^r(\mathcal{H}^{\text{et}}_{\mathcal{M}})$ on S which coincide with $\mathcal{H}_{\mathcal{M}}$, $F^r(\mathcal{H}_{\mathcal{M}})$ over $\mathbf{Q}_p$. Moreover, by elementary commutative algebra (cf., e.g., [Mzk1], Chapter VI, Lemma 1.1), $F^r(\mathcal{H}^{\text{et}}_{\mathcal{M}})$ is necessarily a *vector bundle of rank* r. Thus, we obtain a morphism

$$\psi_r : (F^{r+1}/F^r)(\mathcal{H}^{\text{et}}_{\mathcal{M}}) \to \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{M})$$

(which, when it is necessary to stress the dependence of ψ_r on $\mathcal{M}$, we shall also write "$\psi_r^{\mathcal{M}}$"). Note that the domain of ψ_r is necessarily a *torsion-free coherent sheaf* on S, so it follows immediately that ψ_r is *injective*. It is not immediately clear, however, that ψ_r is *surjective* (although this would follow if, for instance, we knew that the domain of ψ_r is a line bundle).

Lemma 4.2. *For N sufficiently large, the morphism ψ_r is surjective.*

Proof. If p is *odd*, then this follows from the final statement in Lemma 4.1, together with Theorem 1.3, (I). If $p = 2$, then a more complicated argument is necessary. This argument will be presented in §8.3. ○

We are now ready to state the main result of the present §:

Theorem 4.3. (Étale Integral Structure on the Hodge Torsors) *Let $S^{\log}$ be a fine, $\mathbf{Z}$-flat noetherian log scheme. Let $C^{\log} \to S^{\log}$ be a log elliptic curve. Write $E \subseteq C$ for the open subscheme which is a one-dimensional semi-abelian scheme, and $E^\dagger \to E$ for the universal extension. Suppose, moreover, that $\mathcal{L}$ is a line bundle on C of relative* *degree* 1 *over* S *such that some positive tensor power of* $\mathcal{L}(-[0_C])$ *is trivial. Thus, there is a natural* ω_E*-torsor* $C^{\mathcal{L}} \to C$*, equipped with a natural action of* $E^{\dagger}$ *(cf. §3). Denote by* $\mathcal{R}^{\mathcal{L}}$ *the push-forward of* $\mathcal{O}_{C^{\mathcal{L}}}$ *to* C*, equipped with its natural filtration* $F^r(\mathcal{R}^{\mathcal{L}})$ *(given by considering sections whose torsorial degree is* $< r$*). Then there is a* **natural integral structure** *(functorial in* $S^{\log}$*)*

$$\mathcal{R}^{\mathcal{L},\mathrm{et}} \subseteq \mathcal{R}^{\mathcal{L}}_{\mathbf{Q}}$$

that satisfies the following properties:

(I) (Subquotients) *The integral structure induced on* $(F^{r+1}/F^r)(\mathcal{R}^{\mathcal{L}}) = \tau_E^{\otimes r}|_C$ *by* $F^r(\mathcal{R}^{\mathcal{L},\mathrm{et}})$ *is given by* $\frac{1}{r!} \cdot \tau_E^{\otimes r}|_C$*.*

(II) (Cohomology) *The surjection* $F^r(\mathcal{R}^{\mathcal{L},\mathrm{et}}) \to \frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_C$ *(cf. (I)) induces an isomorphism on cohomology after arbitrary base-change:*

$$\mathbf{R}f^1_*(F^r(\mathcal{R}^{\mathcal{L},\mathrm{et}}) \otimes_{\mathcal{O}_S} \mathcal{O}_T) \cong \mathbf{R}f^1_*(\frac{1}{(r-1)!} \cdot \tau_E^{\otimes r-1}|_C \otimes_{\mathcal{O}_S} \mathcal{O}_T)$$

(where $f : C \to S$ *is the structure morphism, and* $T \to S$ *is an arbitrary — i.e., not necessarily* $\mathbf{Z}$*-flat —* S*-scheme). In particular, we have:* $\mathbf{R}f^1_*(\mathcal{R}^{\mathcal{L},\mathrm{et}} \otimes_{\mathcal{O}_S} \mathcal{O}_T) = 0$*.*

(III) (Global Sections) *We have:* $f_*(\mathcal{R}^{\mathcal{L},\mathrm{et}} \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \mathcal{O}_T$ *(for any* S*-scheme* T*).*

(IV) (Description at Infinity) *When* $S = \mathrm{Spec}(\mathbf{Z}[[q^{\frac{1}{n}}]][q^{-1}])$ *(where* $n \in 2 \cdot \mathbf{Z}_{>0}$*, and* q *is the "q-parameter," defined in a formal neighborhood of the point at infinity of* $\overline{\mathcal{M}}_{1,0}$*), the integral structure* $F^r(\mathcal{R}^{\mathcal{L},\mathrm{et}})$ *is that given by the "*$\mathcal{R}^{\mathrm{et}}_{\chi} \stackrel{\mathrm{def}}{=} \bigoplus_{j \geq 0} \mathcal{O}_{C^{\infty}_{\widehat{S}}} \cdot T_{\chi}^{[j]}$*" (for* $\chi_{\mathcal{L}}$ *corresponding to* $\mathcal{L}$*) discussed in §4.2.*

(V) (Torsor Structure) $\mathcal{R}^{\mathcal{L},\mathrm{et}}$ *is an* $\mathcal{O}_C$*-algebra which is compatible with the action of* $E^{\dagger}$ *on* $C^{\mathcal{L}}$*.*

Moreover, the resulting action of $E^{\dagger}$ *on* $C^{\mathcal{L}}_{\mathrm{et}} \stackrel{\mathrm{def}}{=} Spec(\mathcal{R}^{\mathcal{L},\mathrm{et}}) \to C$ *extends to an action of* $E^{\dagger}_{\mathrm{et}}$ *on* $C^{\mathcal{L}}_{\mathrm{et}}$ *which defines a structure of* $E^{\dagger}_{\mathrm{et}}$**-torsor** *on* $E^{\mathcal{L}}_{\mathrm{et}} \stackrel{\mathrm{def}}{=} C^{\mathcal{L}}_{\mathrm{et}}|_E$*.*

Proof. First, observe that by the isomorphisms in the first paragraph of the statement of Proposition 3.4, it suffices to prove Theorem 4.3 in the case $\mathcal{L} = \mathcal{O}_C(0_C)$. Then the existence of $C^{\mathcal{L}}_{\mathrm{et}}$ follows by applying "Proj" to Lemma 4.2. Moreover, Property (I) (respectively, (IV); (V)) follows from Lemma 4.2 (respectively, the construction of $C^{\mathcal{L}}_{\mathrm{et}}$ near infinity in §4.2; by checking that (V) holds near infinity). Properties (II) and (III) may be checked as in §1 in the case of $E^{\dagger}_C$ (by using the analogues in the present context of

Lemma 1.2 and the various long exact sequences that appeared in §1). The final assertions concerning the action of $E^\dagger$ may be checked near infinity. This completes the proof. $\bigcirc$

Definition 4.4. In the context of Theorem 4.3, if $\mathcal{N}$ is a line bundle on C of relative degree d such that some positive tensor power of $\mathcal{N}(-d \cdot [0_C])$ is trivial, then we shall say that $\mathcal{N}$ is *of torsion type*.

Finally, before proceeding, we introduce one more notion, which we will use in §8.3. Let $N \geq 1$ be an integer. Then in addition to the integral structures $E^\dagger_{C,\mathrm{et}}$, $C^{\mathcal{L}}_{\mathrm{et}}$ defined by the algebras $\mathcal{R}^{\mathrm{et}}$, $\mathcal{R}^{\mathcal{L},\mathrm{et}}$ in Theorems 1.3, 4.3, we also have *intermediate étale integral structures* $E^{\dagger;\{N\}}_{C,\mathrm{et}}$, $C^{\mathcal{L};\{N\}}_{\mathrm{et}}$ defined by forming the spectra of the algebras

$$\mathcal{R}^{\mathrm{et};\{N\}} \subseteq \mathcal{R}^{\mathrm{et}}; \quad \mathcal{R}^{\mathcal{L},\mathrm{et};\{N\}} \subseteq \mathcal{R}^{\mathcal{L},\mathrm{et}}$$

where $\mathcal{R}^{\mathrm{et};\{N\}}$ (respectively, $\mathcal{R}^{\mathcal{L},\mathrm{et};\{N\}}$) is the subalgebra of $\mathcal{R}^{\mathrm{et}}$ (respectively, $\mathcal{R}^{\mathcal{L},\mathrm{et}}$) generated by $F^{N+1}(\mathcal{R}^{\mathrm{et}})$ (respectively, $F^{N+1}(\mathcal{R}^{\mathrm{et}})$). Note that the subquotients $(F^{r+1}/F^r)(-)$ of these intermediate integral structures are given by

$$\frac{1}{\gamma(N,r)} \cdot \tau_E^{\otimes r}|_C$$

(cf. property (I) of Theorems 1.3, 4.3), where $\gamma(N,r)$ is the least common multiple of the set of integers

$$c_1! \cdot c_2! \cdot \ldots \cdot c_a!$$

where the $c_1, c_2, \ldots, c_a \in \{1, 2, 3, \ldots, N\}$ satisfy $c_1 + c_2 + \ldots + c_a = r$.

§5. Construction of the Connection

In this §, we construct (for a given log elliptic curve $C^{\log} \to S^{\log}$) a *connection on the pair* $(E^*_{\mathrm{et}}, \mathcal{L}|_{E^*_{\mathrm{et}}})$, where "$*$" is either $\dagger$, or $\mathcal{N}$ (for some degree one line bundle $\mathcal{N}$ of torsion type on C), and $\mathcal{L}|_{E^*_{\mathrm{et}}}$ is the pull-back to E^*_{et} of a line bundle $\mathcal{L}$ on C. It is the elementary study of the properties of this connection that is the main purpose of the present paper. The main technical tool that allows us to construct this connection is properties (II) and (III) of Theorems 1.3, 4.3. We begin the discussion of this § by motivating our construction in the abstract algebraic case by first examining the complex analogue of the abstract algebraic theory.

§5.1. Complex Analogue:

Let E be an *elliptic curve over* $\mathbf{C}$ (the field of complex numbers). In this discussion of the complex analogue, we shall regard E as a *complex manifold* (rather than an algebraic variety). Let us write $\mathcal{O}_E$ (respectively, $\mathcal{O}_{E_{\mathbf{R}}}$) for the sheaf of complex analytic (respectively, real analytic) complex-valued functions on E. In both the complex and real analytic categories, we have *exponential exact sequences*:

$$0 \longrightarrow 2\pi i \cdot \mathbf{Z} \longrightarrow \mathcal{O}_E \xrightarrow{\exp} \mathcal{O}_E^\times \longrightarrow 0$$

$$0 \longrightarrow 2\pi i \cdot \mathbf{Z} \longrightarrow \mathcal{O}_{E_{\mathbf{R}}} \xrightarrow{\exp} \mathcal{O}_{E_{\mathbf{R}}}^\times \longrightarrow 0$$

Since (as is well-known from analysis) $H^1(E, \mathcal{O}_{E_{\mathbf{R}}}) = H^2(E, \mathcal{O}_{E_{\mathbf{R}}}) = H^2(E, \mathcal{O}_E) = 0$, taking cohomology thus gives rise to the following exact sequences:

$$0 \longrightarrow H^1(E, 2\pi i \cdot \mathbf{Z}) \longrightarrow H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \mathcal{O}_E^\times) \xrightarrow{\deg} H^2(E, 2\pi i \cdot \mathbf{Z}) = \mathbf{Z} \longrightarrow 0$$

$$0 \to H^1(E, \mathcal{O}_{E_{\mathbf{R}}}^\times) \xrightarrow{\deg} H^2(E, 2\pi i \cdot \mathbf{Z}) = \mathbf{Z} \to 0$$

In other words, (as is well-known) the isomorphism class of a *holomorphic line bundle* on E is not determined just by its degree (which is a topological invariant), but instead also has continuous holomorphic moduli (given by $H^1(E, \mathcal{O}_E) \neq 0$), while the isomorphism class of *real analytic line bundle* is completely determined by its degree. Thus, in particular, *the complex analytic pair* $(E, \mathcal{L})$ *(i.e., a "polarized elliptic curve") has nontrivial moduli*, and in fact, even if the moduli of E are held fixed, $\mathcal{L}$ itself has nontrivial moduli. (Here, by "nontrivial moduli," we mean that there exist continuous families of such objects which are not locally isomorphic to the trivial family.) On the other hand, (if we write $\mathcal{L}_{\mathbf{R}} \stackrel{\mathrm{def}}{=} \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbf{R}}}$, then) *the real analytic pair* $(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ *has trivial moduli*, i.e., continuous families of such objects are always locally isomorphic to the trivial family. Put another way,

> *The real analytic pair* $(E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ *is a* **topological invariant** *of the polarized elliptic curve* $(E, \mathcal{L})$.

Note that once one admits that $E_{\mathbf{R}}$ itself is a "topological invariant" of E (a fact which may be seen immediately by thinking of $E_{\mathbf{R}}$ as $H^1(E, \mathbf{S^1})$, where $\mathbf{S^1} \subseteq \mathbf{C}^\times$ is the unit circle), (one checks easily that) the fact that $\mathcal{L}_{\mathbf{R}}$ is also a topological invariant follows essentially from the fact that $H^1(E, \mathcal{O}_{E_{\mathbf{R}}}) = 0$. Note that if one thinks of $E^\dagger_{\mathrm{et}}$ as the *algebraic analogue* of $E_{\mathbf{R}}$, then the analogue of this fact in the algebraic context is given precisely by Theorem 1.3, (II). In the present §, we would like to exploit this fact to show that the pair $(E^{\dagger}_{\mathrm{et}}, \mathcal{L}|_{E^{\dagger}_{\mathrm{et}}})$ (or, more generally, $(E^{*}_{\mathrm{et}}, \mathcal{L}|_{E^{*}_{\mathrm{et}}})$) has a natural structure of *crystal* (valued in the category of "polarized varieties," or varieties equipped with an ample line bundle). Thus, in summary, the analogy that we wish to assert here is the following:

$$\textbf{topological}\ \text{invariance of}\ (E_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}}) \ \longleftrightarrow\ (E^{\dagger}_{\mathrm{et}}, \mathcal{L}|_{E^{\dagger}_{\mathrm{et}}})\ \text{is a}\ \textbf{crystal}$$

§5.2. The Schematic Case:

Now let us return to the abstract algebraic context. Let

$$C^{\log} \to S^{\log}$$

be a *log elliptic curve* as in §1. Moreover, let us assume that we are given a *fine log scheme* $T^{\log}$, together with a morphism $S^{\log} \to T^{\log}$, such that S is T-*flat*, and T is $\mathbf{Z}$-*flat*. Then we shall write

$$S^{\log} \times^{\mathrm{PD},\infty}_{T^{\log}} S^{\log}$$

for the (ind-log scheme) given by taking the *formal divided power envelope of the diagonal* inside $S^{\log} \times_{T^{\log}} S^{\log}$. Similarly, for $n \geq 0$ an integer, we shall write

$$S^{\log} \times^{\mathrm{PD},n}_{T^{\log}} S^{\log}$$

for the *n-th PD-infinitesimal neighborhood of the diagonal* in $S^{\log} \times^{\mathrm{PD},\infty}_{T^{\log}} S^{\log}$. (That is to say, if the diagonal is defined by a PD-ideal $\mathcal{I}$, then $S^{\log} \times^{\mathrm{PD},n}_{T^{\log}} S^{\log}$ is defined by the n-th divided power of $\mathcal{I}$.) In the following discussion, $T^{\log}$ will be *fixed*, so we will *omit it, in order to simplify the notation.*

Let us write $\pi_1, \pi_2 : S^{\log} \times^{\mathrm{PD},\infty} S^{\log} \to S^{\log}$ for the *left and right projections*, respectively. Note that it follows immediately from the definition of the universal extension as a parameter space for line bundles equipped with a connection (cf. [Mzk1], Chapter III, §1), that $E^{\dagger} \to S$ is equipped with a structure of *log crystal* over $S^{\log}$. This structure of log crystal on $E^{\dagger} \to S$ gives rise to a natural isomorphism:

$$\Xi_{E^{\dagger}} : \pi_1^*(E^{\dagger}) \cong \pi_2^*(E^{\dagger})$$

between the two pull-backs of $E^{\dagger} \to S$ to $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$. Moreover, this isomorphism satisfies the "cocycle condition" with respect to the various pull-backs to the triple product $S^{\log} \times^{\mathrm{PD},\infty} S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$. In fact, by considering the universal case (i.e., the case when S is étale over $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$), if follows from the fact $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ is regular of dimension 2 (together with the fact that $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$ is S-flat) that this isomorphism extends to an *isomorphism between the compactifications*

$$\Xi_{E_C^\dagger} : \pi_1^*(E_C^\dagger) \cong \pi_2^*(E_C^\dagger)$$

(cf. [Mzk1], Chapter III, §4).

Now suppose that $S^{\log}$ is *log smooth* over $T^{\log}$. Then it follows that $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$ is S-flat, hence, in particular, $\mathbf{Z}$-flat, so *it makes sense to speak of integral structures on flat objects over* $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$. Now if $\mathcal{N}$ is a *line bundle* on C *of torsion type* (cf. Definition 4.4), then there is a natural *identification*

$$C^{\mathcal{N}} \otimes \mathbf{Q} = E_C^\dagger \otimes \mathbf{Q}$$

(cf. the first Remark following Proposition 3.4). That is to say, $C^{\mathcal{N}}$ may be regarded as "another integral structure" on $E_C^\dagger$. Similarly, the étale integral structure $E_{C,\mathrm{et}}^\dagger$ (cf. Theorem 1.3), as well as the étale integral structure $C_{\mathrm{et}}^{\mathcal{N}}$ (cf. Theorem 4.3), may be regarded as "integral structures on $E_C^\dagger$."

> When we wish to discuss these integral structures on a similar footing, we shall use the notation "$*$," where $* \in \{\dagger, \mathcal{N}\}$.

Thus, if $* = \dagger$, then $E_C^* = E_C^\dagger$, $E_{C,\mathrm{et}}^* = E_{C,\mathrm{et}}^\dagger$, while if $* = \mathcal{N}$, then $E_C^* = C^{\mathcal{N}}$, $E_{C,\mathrm{et}}^* = C_{\mathrm{et}}^{\mathcal{N}}$.

Lemma 5.1. *Let $* \in \{\dagger, \mathcal{N}\}$. If $* = \mathcal{N}$, then let us suppose that the following condition is satisfied:*

> $(*^{\mathrm{KS}})$ *The derivative*
>
> $$\Omega_{(\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}/\mathbf{Z}}|_{S^{\log}} \to \Omega_{S^{\log}/T^{\log}}$$
>
> *of the classifying morphism $S^{\log} \to (\overline{\mathcal{M}}_{1,0}^{\log})_{\mathbf{Z}}$ of $C^{\log} \to S^{\log}$ (i.e., the "Kodaira-Spencer morphism" of the family $C^{\log} \to S^{\log}$) factors through $2 \cdot \Omega_{S^{\log}/T^{\log}}$.*

Then the isomorphism $\Xi_{E_C^\dagger}$ is compatible with the integral structures of E_C^*, $E_{C,\mathrm{et}}^*$.

Remark. In fact, in the case of $E_{C,\mathrm{et}}^*$, the assumption $(*^{\mathrm{KS}})$ may be *omitted.* This is the topic of Corollary 8.3, to be proven in §8.1. The reader may check easily that there are no "vicious circles" in the reasoning.

Proof. First, we make the following elementary reductions:

(1) By the elementary theory of integrable connections, it suffices to prove the result over $S^{\log} \times^{\mathrm{PD},2} S^{\log}$ (as opposed to $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$).

(2) *Away from infinity*, by *translating* by the unique torsion point $\alpha^{\dagger} \in E^{\dagger}(S \otimes \mathbf{Q})$ that lifts the torsion point α defined by $\mathcal{N}(-0_E)$ (cf. Proposition 3.4), we can reduce to the case where $\mathcal{N} = \mathcal{O}_C(0_C)$. Note that here, we use the fact that, at least in characteristic zero (i.e., after tensoring with $\mathbf{Q}$) the section $\alpha^{\dagger}$ of $E^{\dagger}$ is necessarily *horizontal* (since it is a torsion point of $E^{\dagger}$).

(3) *Near infinity*, we can reduce to the *universal case*, e.g., the case in which S is smooth of dimension 1 over a ring of the form $T^{\log} = \mathrm{Spec}(\mathbf{Z}[\zeta])$ (where ζ is a root of unity, and $T^{\log}$ is equipped with the trivial log structure), and the q-parameter of the family $C^{\log} \to S^{\log}$ defines a $\mathbf{Z}$-flat Cartier divisor D on S. Since such an S is regular of dimension 2, integral structures over S are determined by their restrictions to $S \backslash D$.

(4) By combining reduction steps (2) and (3), we thus reduce to the *universal case away from infinity*, i.e., where $\mathcal{N} = \mathcal{O}_E(0_E)$; $T = \mathrm{Spec}(\mathbf{Z})$; S is étale over $(\mathcal{M}_{1,0})_{\mathbf{Z}}$; and S and T are equipped with the trivial log structure. In fact, we may also assume that the reduction of $E \to S$ modulo 2 is a family of *ordinary* elliptic curves.

(5) By replacing the S of (4) by a suitable double cover, we may assume that there exists a 2-torsion point $\alpha \in E(S)$ such that the $E^{\mathcal{N}}$ for $\mathcal{N} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(\alpha)$ *coincides* (as an integral structure) with $E^{\dagger}$. (Indeed, in the notation of 4.2, we choose an α such that the resulting "i_{χ}/n" is $\frac{1}{2}$.) Thus, by translating as in (2) (to make up for the difference between the $\mathcal{N} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(0_E)$ of (4) and the $\mathcal{N} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(\alpha)$ of the present reduction step), *we reduce to the case where* $* = \dagger$. (Note that passing to a double cover as above means that *even in the universal case*, we are, in effect, working with the differentials "$\frac{1}{2} \cdot \Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}/\mathbf{Z}}$" — i.e., not with the differentials $\Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}/\mathbf{Z}}$. This is what necessitates the assumption ($*^{\mathrm{KS}}$) in the case $* = \mathcal{N}$. Also note that the fact that it suffices to look only at the *differentials* (as opposed to $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$ in its entirety) follows from reduction step (1).)

Thus, in summary, *it suffices to prove the result in the case of* $* = \dagger$*,* S *étale over* $(\mathcal{M}_{1,0})_{\mathbf{Z}}$.

In this case, however, the integral structure of $E^{\dagger}_{\mathrm{et}}$ is obtained near infinity by adjoining the "$T^{[r]}$" (cf. §1). Thus, it suffices to observe that the indeterminate "T" of the discussion of §1 is the same, whether pulled back by π_1 or π_2. This will follow as soon as we verify that the canonical section $\kappa_{\widehat{S}}$ of §1, as well as the invariant differential $d\log(U)$ satisfy this property (i.e., are horizontal). But the horizontality of $\kappa_{\widehat{S}}$ follows from the fact that it may be characterized as the unique section which is a *group homomorphism.* The horizontality of the differential $d\log(U)$ follows from that of the multiplicative coordinate U, which, in turn, follows from the well-known fact that automorphisms of group schemes of multicative type (i.e., such as $\mathbf{G}_{\mathrm{m}}$) are *rigid.* This completes the proof of Lemma 5.1. $\bigcirc$

Remark. Note that, at least in the case $* = \dagger$, by pulling back from $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, one thus obtains a structure of *log crystal* on $E^*_{C,\mathrm{et}}$ — *valid even for bases S which are not log smooth over some $T^{\log}$ (where T is $\mathbf{Z}$-flat).*

Theorem 5.2. (The Universal Extension as a Polarized Log Crystal) *Let $* \in \{\dagger, \mathcal{N}\}$, where $(*^{\mathrm{KS}})$ is to be satisfied if $* = \mathcal{N}$ (cf. Lemma 5.1). Then the étale integral structure on E^*_C (cf. Theorems 1.3, 4.3) defines an S-scheme $E^*_{C,\mathrm{et}} \to S$, which is equipped with a natural structure of log crystal with respect to the morphism $S^{\log} \to T^{\log}$, i.e., we have a natural isomorphism:*

$$\Xi_{E^*_{C,\mathrm{et}}} : \pi_1^*(E^*_{C,\mathrm{et}}) \cong \pi_2^*(E^*_{C,\mathrm{et}})$$

which satisfies the "cocycle condition" with respect to the various pull-backs to the triple product $S^{\log} \times^{\mathrm{PD},\infty} S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$. Moreover, if $\mathcal{L}$ is any **line bundle** *on $E^*_{C,\mathrm{et}}$, and $\sigma : S \to E^*_{C,\mathrm{et}}$ is a* **horizontal section of** *$E^*_{C,\mathrm{et}}$, then restriction to σ defines a bijection between isomorphisms*

$$\pi_1^*\mathcal{L} \cong \pi_2^*\mathcal{L}$$

(of line bundles on $\pi_1^*(E^*_{C,\mathrm{et}}) \cong \pi_2^*(E^*_{C,\mathrm{et}})$) and isomorphisms

$$\pi_1^*\mathcal{L}_\sigma \cong \pi_2^*\mathcal{L}_\sigma$$

(of line bundles on $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$). Here, we write $\mathcal{L}_\sigma \stackrel{\mathrm{def}}{=} \sigma^*\mathcal{L}$. Moreover, the same assertion holds if we replace "$\times^{\mathrm{PD},\infty}$" by "$\times^{\mathrm{PD},n}$" for any (finite) integer $n \geq 1$.

Proof. It remains only to prove the asserted bijection between isomorphisms of line bundles. Clearly, this assertion is *local on S* (with respect to, say, the étale topology on S). Thus, we may assume that S is *affine.* In this case, the asserted bijection will follow as soon as we show that there exists at least one isomorphism $\pi_1^*\mathcal{L} \cong \pi_2^*\mathcal{L}$, and that any other isomorphism is obtained by multiplication by an invertible function on $S^{\log} \times^{\mathrm{PD},\infty} S^{\log}$. But since deformations of such line bundles (respectively, deformations of such isomorphisms of line bundles) are parametrized by H^1 (respectively, H^0) of the structure sheaf $\mathcal{O}_{E^*_{C,\mathrm{et}}}$, we thus see that the desired assertions follow from properties (II), (III) of Theorems 1.3, 4.3. $\bigcirc$

Remark. If $* = \dagger$ (respectively, $* = \mathcal{O}_C(\alpha)$, for a torsion point $\alpha \in E(S)$), then a *natural choice* for the horizontal section σ is the *zero section* of $E^\dagger_{C,\mathrm{et}}$ (respectively, section of $E^*_{C,\mathrm{et}}$

determined by the *torsion lifting* $\alpha^\dagger \in E^\dagger_{C,\mathrm{et}}(S \otimes \mathbf{Q})$ as in Proposition 3.4). Thus, if one takes $\mathcal{L}$ to be an *ample line bundle* and fixes a *rigidification*

$$\mathcal{L}_\sigma \cong \mathcal{O}_S$$

(so $\mathcal{L}_\sigma$ gets a structure of log crystal arising from the trivial structure of log crystal on $\mathcal{O}_S$) then this rigidification defines (by the bijection of Theorem 5.2) a structure of *"log crystal valued in the category of polarized varieties"* on the pair

$$(E^*_{C,\mathrm{et}}, \mathcal{L}|_{E^*_{C,\mathrm{et}}})$$

(where, if one replaces $E^*_{C,\mathrm{et}}$ by E^*_{et}, then one may replace "varieties" by "smooth group schemes" (respectively, "$E^\dagger_{\mathrm{et}}$-torsors") when $* = \dagger$ (respectively, $* = \mathcal{N}$)). Put another way, this rigidification defines a natural *integrable logarithmic connection* (relative to the morphism $S^{\log} \to T^{\log}$) on the pair $(E^*_{C,\mathrm{et}}, \mathcal{L}|_{E^*_{C,\mathrm{et}}})$. Finally, we remark that *the construction and demonstration of the elementary properties of this sort of connection constitute the main goal of the present paper.*

Remark. Note that although we used the étale integral structure on E^*_C in order to define a connection on the pair $(E^*_{C,\mathrm{et}}, \mathcal{L}|_{E^*_{C,\mathrm{et}}})$, in fact, it is not difficult to prove (cf. the discussion of "Griffiths semi-transversality" in §8.1) that (except at the prime $p = 2$) *this connection arises from a connection on the pair* $(E^*_C, \mathcal{L}|_{E^*_C})$ (i.e., equipped with the usual integral structure). We believe, however, that because the crucial properties (II), (III) of Theorems 1.3, 4.3, do not hold for the usual integral structure, it is nonetheless *much more natural to consider this connection in the context of the étale integral structure.* Another reason for this is that the important "Schottky-Theoretic Hodge-Arakelov Comparison Isomorphism" of §6, as well as the related vanishing of the higher p-curvatures discussed in §7, only hold with respect to the étale integral structure.

Theorem 5.3. (Functoriality) *Suppose that the subscheme $D \subseteq S$ (i.e., the pull-back of the divisor at infinity of $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$) forms a $\mathbf{Z}$-flat Cartier divisor on S. Suppose that $\widetilde{C}^{\log} \to S^{\log}$ is another log elliptic curve, and that we are given an* **isogeny**

$$\widetilde{E} \to E$$

(which implies that the subscheme "$\widetilde{D}$" corresponding to $\widetilde{E}$ also forms a $\mathbf{Z}$-flat Cartier divisor on S). Let $* \in \{\dagger, \{\widetilde{\mathcal{N}}, \mathcal{N}\}\}$, where $(*^{\mathrm{KS}})$ is to be satisfied by both E and $\widetilde{E}$ if $* = \{\widetilde{\mathcal{N}}, \mathcal{N}\}$ (cf. Lemma 5.1). Here, we assume that $\widetilde{\mathcal{N}}$ is a degree one torsion type line bundle on $\widetilde{C}$ whose push-forward to C (via $\widetilde{E} \to E$) is equal to $\mathcal{N}$. In the following, when $* = \{\widetilde{\mathcal{N}}, \mathcal{N}\}$, "$*$" is to be interpreted as $\widetilde{\mathcal{N}}$ (respectively, $\mathcal{N}$) when it appears on tilded (respectively, non-tilded) objects. Then by pushing forward line bundles with connection

on $\widetilde{E}$ to E (cf. the functorial definition of the universal extension discussed in [Mzk1], Chapter III, §1; cf. also the discussion of [Mzk1], Chapter IV, §3), this isogeny defines morphisms

$$\widetilde{E}^* \to E^*$$

which are **compatible with the étale integral structures**, *hence give rise to morphisms $\widetilde{E}^*_{\mathrm{et}} \to E^*_{\mathrm{et}}$. Finally, if $\mathcal{L}$ is a line bundle on E^*_C (whose pull-back to $\widetilde{E}^*_C$ we denote by $\widetilde{\mathcal{L}}$), and $\sigma \in E^*_{C,\mathrm{et}}(S)$, $\widetilde{\sigma} \in \widetilde{E}^*_{C,\mathrm{et}}(S)$ are compatible horizontal sections, then the bijections (for $\mathcal{L}$, $\widetilde{\mathcal{L}}$) between isomorphisms of line bundles appearing in Theorem 5.2 are compatible with the morphism $\widetilde{E}^*_{\mathrm{et}} \to E^*_{\mathrm{et}}$.*

Proof. First, note that the push-forward morphism $\widetilde{E}^* \to E^*$ is clearly defined over $\mathbf{Q}$; to see that it is integral over $\mathbf{Z}$, one first reduces to the case $* = \dagger$ by translating as in the proof of Lemma 5.1. In the case $* = \dagger$, it suffices to observe this integrality (at a prime p) for line bundles whose l-th power is trivial (where $(l, p) = 1$). But this integrality follows immediately (by raising to the l-th power) for such line bundles from the corresponding integrality for the trivial line bundle. Moreover, this integrality for the trivial line bundle follows from the elementary fact that the *trace of a differential* on $\widetilde{E}$ gives rise to an *integral* differential on E. This shows that $\widetilde{E}^* \to E^*$ is integral.

Since $\widetilde{E}^\dagger \to E^\dagger$ is clearly a *group scheme homomorphism*, it follows that it is compatible with the respective canonical sections "$\kappa_{\widehat{S}}$" (cf. the discussion of §1; the proof of Lemma 5.1). Thus, we conclude the desired integrality of $\widetilde{E}^\dagger_{\mathrm{et}} \to E^\dagger_{\mathrm{et}}$. More generally, the integrality of $\widetilde{E}^*_{\mathrm{et}} \to E^*_{\mathrm{et}}$ follows by translating as in the proof of Lemma 5.1.

The compatibility with the bijections between isomorphisms of line bundles appearing in Theorem 5.2 then follows immediately from the facts that: (i) $\widetilde{E}^*_{\mathrm{et}} \to E^*_{\mathrm{et}}$ maps $\widetilde{\sigma}$ to σ; (ii) because of our assumption on D, it suffices to carry out the verification of the asserted compatiblity on $S \backslash D = S \backslash \widetilde{D}$ (over which E^*_{et}, $\widetilde{E}^*_{\mathrm{et}}$ coincide with $E^*_{C,\mathrm{et}}$, $\widetilde{E}^*_{C,\mathrm{et}}$, respectively). This completes the proof. $\bigcirc$

Remark. Note in particular that Theorem 5.3 applies to the case where $\widetilde{E} \to E$ is given by an *automorphism of* E, such as the automorphism defined by multiplication by -1 (which we denote by $[-1]$). Thus, if the line bundles $\mathcal{N}$, $\mathcal{L}$, together with the *rigidification* of $\mathcal{L}$ (cf. the Remark following Theorem 5.2), are fixed by $[-1]$ (i.e., are *symmetric*), then we get a natural action of ± 1 on the resulting *log crystal (structure on)* $(E^*_{C,\mathrm{et}}, \mathcal{L}|_{E^*_{C,\mathrm{et}}})$.

§6. The Schottky-Theoretic Hodge-Arakelov Comparison Isomorphism

In this §, we prove a *simplified version of the "Hodge-Arakelov Comparison Isomorphism"* of [Mzk1], in a neighborhood of infinity. We then apply this *Schottky-Theoretic*

Comparison Isomorphism to prove various technical results concerning this object, e.g., that *its p-curvature is zero.* This property is particularly remarkable in that typically, for $\mathcal{MF}^\nabla$-objects as in [Falt], §2, *zero p-curvature is related to vanishing of the Kodaira-Spencer morphism* (cf. [Katz1,2]). On the other hand, in §8, we shall see that the analogue of the Kodaira-Spencer morphism in the present context *does not vanish.* Thus, we see (even without the aid of a Frobenius action) that the polarized varieties with connection constructed in the present paper possess properties that are *intrinsically different* from the $\mathcal{MF}^\nabla$-objects of [Falt], §2. We will study properties related to the p-curvature in more detail in §7. In addition, (in the present §) we compute the *monodromy* of the object with connection under consideration in a neighborhood of infinity.

In this §, we use the notation of §4.2. *The first goal of our discussion is to describe the connection constructed in Theorem 5.2 explicitly using U, θ^m.* To do this, let us first observe that the *canonical section "κ"* (of [Mzk1], Chapter III, Theorem 2.1) defines a natural morphism

$$\kappa^{\mathrm{et}}_{C^\infty_{\widehat{S}}} : C^\infty_{\widehat{S}} \to (E^\dagger_{C,\mathrm{et}})_{\widehat{S}}$$

(cf. [Mzk1], Chapter III, Theorem 5.6, as well as the section "$\kappa_{\widehat{S}}$" and Theorem 1.3, (IV), of §1 of the present paper). If we compose this morphism with the natural projection $\widetilde{C}^\infty_{\widehat{S}} \to C^\infty_{\widehat{S}}$, then we obtain a morphism

$$\kappa^{\mathrm{et}}_{\widetilde{C}^\infty_{\widehat{S}}} : \widetilde{C}^\infty_{\widehat{S}} \to (E^\dagger_{C,\mathrm{et}})_{\widehat{S}}$$

Note that $\kappa^{\mathrm{et}}_{C^\infty_{\widehat{S}}}$ is *horizontal* with respect to the natural connection on $E^\dagger_{C,\mathrm{et}}$ (cf. Theorem 5.2) and the connection on $\widetilde{C}^\infty_{\widehat{S}}$ arising from the description of $\widetilde{C}^\infty_{\widehat{S}}$ as the pull-back to $\widehat{S}$ of the "Néron model" of $(\mathbf{G}_{\mathrm{m}})_{\mathcal{O}[[q]][q^{-1}]}$ over $\mathcal{O}[[q]]$.

In the following discussion, let us set:

$$* \stackrel{\mathrm{def}}{=} \mathcal{L}^\chi$$

Note that *if we tensor with* $\mathbf{Q}$, then $(E^\dagger_{C,\mathrm{et}})_{\widehat{S}} \otimes \mathbf{Q} = (E^*_{C,\mathrm{et}})_{\widehat{S}} \otimes \mathbf{Q}$. In particular, over $\mathbf{Q}$, we may regard $\kappa^{\mathrm{et}}_{C^\infty_{\widehat{S}}}$ as a morphism to $(E^*_{C,\mathrm{et}})_{\widehat{S}}$. Also, over $\mathbf{Q}$, the zero section of $(E^\dagger_{C,\mathrm{et}})_{\widehat{S}}$ defines a horizontal section $\epsilon \in (E^*_{C,\mathrm{et}})_{\widehat{S}}(\widehat{S} \otimes \mathbf{Q})$. Now we have the following:

Theorem 6.1. (Explicit Description of the Connection) *Denote by ∇^{alg} the connection on $(\widetilde{C}^\infty_{\widehat{S}}, \mathcal{L}^\chi_{\widetilde{C}^\infty_{\widehat{S}}}) \otimes \mathbf{Q}$ obtained by pulling back via $\kappa^{\mathrm{et}}_{C^\infty_{\widehat{S}}}$ the connection on $(E^*_{C,\mathrm{et}}, \mathcal{L}^\chi_{E^*_{C,\mathrm{et}}})$*

determined by some **rigidification** *(cf. Theorem 5.2) at ϵ. On the other hand, denote by ∇^∞ the connection on $(\widetilde{C}^\infty_{\widehat{S}}, \mathcal{L}^\chi_{\widetilde{C}^\infty_{\widehat{S}}})$ obtained by declaring the section $\widetilde{U}^{i_\chi} \cdot \theta^m$ to be horizontal. Then, if we think of ∇^{alg} and ∇^∞ as logarithmic connections on the object $\Gamma(C^\infty_{\widehat{S}}, \mathcal{L}^\chi_{C^\infty_{\widehat{S}}}) \otimes \mathbf{Q}$ over $S^{\log}$, then both ∇^{alg} and ∇^∞ are* **integral** *(i.e., defined without tensoring with $\mathbf{Q}$), and, moreover, we have:*

$$\nabla^{\mathrm{alg}} = \nabla^\infty - \omega_\infty$$

where $\omega_\infty \stackrel{\mathrm{def}}{=} \frac{d\log(\mathfrak{F})}{dq} \cdot dq$, and $\mathfrak{F} \in A^\times$ depends on the rigidification chosen. In particular, these two connections agree up to a **scalar** *differential form.*

Proof. Indeed, the point here (cf. the proof of Theorem 5.2) is that the difference between the two connections on the pair $(\widetilde{C}^\infty_{\widehat{S}}, \mathcal{L}^\chi_{\widetilde{C}^\infty_{\widehat{S}}}) \otimes \mathbf{Q}$ naturally forms a section

$$\in \Gamma(\widetilde{C}^\infty_{\widehat{S}}, \mathcal{O}_{\widetilde{C}^\infty_{\widehat{S}}} \otimes \mathbf{Q}) \cdot d\log(q) = A_{\mathbf{Q}} \cdot d\log(q)$$

(where $A_{\mathbf{Q}} \stackrel{\mathrm{def}}{=} A \otimes \mathbf{Q}$), i.e., this difference is a *constant* (relative to the morphism $\widetilde{C}^\infty_{\widehat{S}} \to \widehat{S}$) — cf. the application of Theorem 1.3, (III), in the proof of Theorem 5.2. (Note that $\Gamma(\widetilde{C}^\infty_{\widehat{S}}, \mathcal{O}_{\widetilde{C}^\infty_{\widehat{S}}} \otimes \mathbf{Q}) = A_{\mathbf{Q}}$ follows, for instance, from the fact that the special fiber of $\widetilde{C}^\infty_{\widehat{S}} \to \widehat{S}$ is an infinite chain of $\mathbf{P}^1$'s.) Thus, it suffices to show that the difference between the connections induced by ∇^{alg} and ∇^∞ on $\mathcal{L}_C|_\epsilon = \mathcal{L}_{\widetilde{C}^\infty_{\widehat{S}}}|_\epsilon$ is as asserted. But note that ∇^∞ is integral by *definition*, and, moreover, has the property that its restriction to $\mathcal{L}_C|_\epsilon$ is *induced by a trivialization of* $\mathcal{L}_C|_\epsilon$. On the other hand, ∇^{alg} is determined by a trivialization of $\mathcal{L}_C|_\epsilon \otimes \mathbf{Q}$. Thus, since these two trivialization differ by a factor $\mathfrak{F} \in A^\times_{\mathbf{Q}}$, ∇^{alg} and ∇^∞ differ by the scalar differential form $\omega_\infty \stackrel{\mathrm{def}}{=} \frac{d\log(\mathfrak{F})}{dq} \cdot dq$. Finally, observe that since $A^\times_{\mathbf{Q}} = A^\times \cdot K^\times$ (where K is the finite extension of $\mathbf{Q}$ which is the quotient field of the ring $\mathcal{O}$ appearing in the definition of A — cf. §4.1), we may take $\mathfrak{F}$ to be in $A^\times$ without affecting ω_∞. This completes the proof. ○

Next, let us write

$$\mathcal{V}_{\mathbf{G}_{\mathrm{m}}} \stackrel{\mathrm{def}}{=} \Gamma((\mathbf{G}_{\mathrm{m}})_{\widehat{S}}, \mathcal{O}_{(\mathbf{G}_{\mathrm{m}})_{\widehat{S}}})$$

for the topological A-module of regular functions on $(\mathbf{G}_{\mathrm{m}})_{\widehat{S}}$. Thus, $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$ is the *free topological A-module on the generators*

$$U^k$$

for $k \in \mathbf{Z}$. On the other hand, we also have

$$(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}} \stackrel{\text{def}}{=} \Gamma((E^*_{C,\text{et}})_{\widehat{S}}, \mathcal{L}^\chi_{(E^*_{C,\text{et}})_{\widehat{S}}})$$

(cf. the notation "$\Gamma(C^{[\eta]}_{\text{et}}, \mathcal{L}^\chi|_{C^{[\eta]}_{\text{et}}})$" used in the discussion at the end of §4.2).

In the following discussion, we would like to consider the *restriction morphism*

$$\Xi : (\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}} \rightarrow \mathcal{V}_{\mathbf{G}_\text{m}}$$

defined by: (i) first, pulling back sections of $\mathcal{L}^\chi_{(E^\dagger_{C,\text{et}})_{\widehat{S}}}$ to $\widetilde{C}^\infty_{\widehat{S}}$ via $\kappa^{\text{et}}_{\widetilde{C}^\infty_{\widehat{S}}} : \widetilde{C}^\infty_{\widehat{S}} \rightarrow (E^\dagger_{C,\text{et}})_{\widehat{S}}$; (ii) applying the trivialization "$\widetilde{U}^{i_\chi} \cdot \theta^m$" discussed above to obtain "usual functions" (i.e., as opposed to sections of a line bundle) on $\widetilde{C}^\infty_{\widehat{S}}$; (iii) observing (cf. the discussion of §4.1) that the resulting functions descend from $\widetilde{C}^\infty_{\widehat{S}}$ to $C^\infty_{\widehat{S}}$ (i.e., may be expressed using integral powers of U, without using $\widetilde{U}$); and, finally, (iv) restricting functions over $C^\infty_{\widehat{S}}$ to $E_{\widehat{S}} = (\mathbf{G}_\text{m})_{\widehat{S}}$. (Observe that, although *a priori*, Ξ is only defined over $\mathbf{Q}$, Ξ is, in fact, *integral* (cf. the discussion of §4.1, 4.2), as the notation suggests.) Note that Ξ is a topological A-morphism of *topological A-modules* which is, moreover, *horizontal up to a scalar differential* (cf. Theorem 6.1) with respect to the natural logarithmic connections (over $S^{\log}$) on both sides. Another way to think of the fact that Ξ is "horizontal up to a scalar differential" is to say that Ξ is *projectively horizontal*, i.e., compatible with the natural *projective connections* on both sides. (We leave the routine formulation of these various "projective notions" to the reader.)

In the following, in order to *analyze* Ξ *in more detail*, we would like to introduce a *natural basis* (i.e., the "ζ^{CG}_r" — cf. the discussion at the end of §4.2) of the domain $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$ of Ξ (following the theory of [Mzk1], Chapter V, §4). In order to discuss this basis, we must first review the relevant notation of [Mzk1], Chapter V, §4. For integers $r \geq 0$, let

$$\lambda_r \stackrel{\text{def}}{=} \left[\frac{r}{2} + \frac{i_\chi}{n}\right]$$

(i.e., the greatest integer $\leq$ the number in brackets), and

$$F^r(\mathbf{Z}) \stackrel{\text{def}}{=} \{0 - \lambda_r, 1 - \lambda_r, \ldots, r - 1 - \lambda_r\} \subseteq \mathbf{Z}$$

Thus, $F^{r+1}(\mathbf{Z}) \supseteq F^r(\mathbf{Z})$ is obtained from $F^r(\mathbf{Z})$ by appending one more integer "$k[r]$" directly to the left/right of $F^r(\mathbf{Z})$ (where "left/right" depends only on the parity of r). In particular, the map $\mathbf{Z}_{\geq 0} \rightarrow \mathbf{Z}$ given by

$$r \mapsto k[r]$$

is a *bijection.* Also, let us write

$$\Psi(k) \stackrel{\mathrm{def}}{=} \frac{1}{2}k^2 + \frac{i_\chi}{n}k$$

(cf. the discussion at the beginning of [Mzk1], Chapter VIII, §3).

Now we are ready to discuss the basis referred to above. This (topological A-) basis of $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$ is given by the *"congruence canonical Schottky-Weierstrass zeta functions"*:

$$\zeta_0^{\mathrm{CG}}, \zeta_1^{\mathrm{CG}}, \ldots, \zeta_r^{\mathrm{CG}}, \ldots \in (\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$$

which are uniquely determined by their images under Ξ:

$$\Xi(\zeta_r^{\mathrm{CG}}) = \sum_{k \in \mathbf{Z}} \binom{k + \lambda_r}{r} \cdot \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{1}{2}k^2 + \frac{i_\chi}{n}k} \cdot U^k$$

(cf. [Mzk1], Chapter V, Theorem 4.8). Moreover, $\zeta_r^{\mathrm{CG}} \equiv 0$ modulo $q^{\Psi(k[r])}$. Thus, we obtain sections

$$\widetilde{\zeta}_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} q^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}} \in q^{-\infty} \cdot (\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$$

In particular, the $\widetilde{\zeta}_r^{\mathrm{CG}}$ define a *new integral structure* on $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$, which we denote by

$$(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}} = \Gamma^{\mathrm{GP}}((E^\dagger_{C,\mathrm{et}})_{\widehat{S}}, \mathcal{L}_{(E^\dagger_{C,\mathrm{et}})_{\widehat{S}}})$$

(where the "GP" stands for *"Gaussian poles,"* i.e., the poles arising from the $q^{-\Psi(k[r])}$ — cf. [Mzk1], Chapter VI, Theorem 4.1, and the Remarks following that theorem). Thus, in particular, the $\widetilde{\zeta}_r^{\mathrm{CG}}$ form a *topological A-basis* for $(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}}$, and Ξ factors through $(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}}$ to form a morphism

$$\Xi^{\mathrm{GP}} : (\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}} \rightarrow \mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$$

Finally, we recall from the theory of [Mzk1] (Chapter V, §4, and Chapter VIII, §3), that the $\Xi^{\mathrm{GP}}(\widetilde{\zeta}_r^{\mathrm{CG}})$ *form a basis of* $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$. Indeed, this follows by observing that, modulo $q^{\frac{1}{n}}$,

$$\Xi^{\mathrm{GP}}(\widetilde{\zeta}_r^{\mathrm{CG}}) \equiv (\in \boldsymbol{\mu}_n) \cdot U^{k[r]} + \ldots$$

where "..." is *either* (depending on the parity of r) zero *or* a term of the form $(\in \boldsymbol{\mu}_n) \cdot U^{k[r+1]}$ — i.e., modulo $q^{\frac{1}{n}}$, it is clear that the $\Xi^{\mathrm{GP}}(\widetilde{\zeta}_r^{\mathrm{CG}})$ form a basis of $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$. In particular, since (by definition) the $\widetilde{\zeta}_r^{\mathrm{CG}}$ form a basis of $(\mathcal{V}_{\mathcal{L}^\chi}^{\mathrm{GP}})_{\widehat{S}}$, we thus obtain that Ξ^{GP} is an *isomorphism.*

Thus, in summary, we obtain the following "Hodge-Arakelov-type Comparison Isomorphism" (i.e., a comparison isomorphism reminiscent of the main theorem of [Mzk1]), except in the present "Schottky-theoretic context" (i.e., the context of the *Schottky uniformization* "$E = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}}$"):

Theorem 6.2. (Schottky-Theoretic Hodge-Arakelov Comparison Isomorphism) *Let $\mathcal{O}$ be a Zariski localization of the ring of integers of a finite extension of $\mathbf{Q}$; $n \stackrel{\mathrm{def}}{=} 2m$;*

$$A \stackrel{\mathrm{def}}{=} \mathcal{O}[[q^{\frac{1}{n}}]]; \quad S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A); \quad \widehat{S} \stackrel{\mathrm{def}}{=} \mathrm{Spf}(A)$$

(where q is an indeterminate, and we regard A as equipped with the q-adic topology). Let $E \to S$ be the **degenerating elliptic curve** *whose Tate parameter is given by q, and $C \to S$ its natural semi-stable compactification. Write $\mathcal{L}_C \stackrel{\mathrm{def}}{=} \mathcal{O}_C(0_C)$ for the line bundle on C defined by the zero section, and assume that we are given a* **character**

$$\chi_{\mathcal{L}} \in \mathrm{Hom}(\mathbf{Z}_{\mathrm{et}} \times \boldsymbol{\mu}_n, \boldsymbol{\mu}_n)$$

(to be thought of as a "twist" applied to $\mathcal{L}_C$). Set $* \stackrel{\mathrm{def}}{=} \mathcal{L}^\chi$, and equip the extension of the universal extension $E^\dagger \to E$ over C with the **étale integral structure for the Hodge torsor corresponding to** *$\mathcal{L}^\chi$ (cf. Theorem 5.2) to form $E^*_{C,\mathrm{et}} \to S$. Let*

$$\mathcal{V}_{\mathbf{G}_{\mathrm{m}}} \stackrel{\mathrm{def}}{=} \Gamma((\mathbf{G}_{\mathrm{m}})_{\widehat{S}}, \mathcal{O}_{(\mathbf{G}_{\mathrm{m}})_{\widehat{S}}}); \quad (\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}} \stackrel{\mathrm{def}}{=} \Gamma((E^*_{C,\mathrm{et}})_{\widehat{S}}, \mathcal{L}_C^\chi|_{(E^*_{C,\mathrm{et}})_{\widehat{S}}})$$

Then pull-back via the **canonical section** *$\kappa^{\mathrm{et}}_{C^\infty_{\widehat{S}}} : C^\infty_{\widehat{S}} \to (E^\dagger_{C,\mathrm{et}})_{\widehat{S}}$ (cf. [Mzk1], Chapter III, Theorem 5.6, as well as the section "$\kappa_{\widehat{S}}$" and Theorem 1.3, (IV), of §1 of the present paper), together with the natural trivialization of $\mathcal{L}_C^\chi$ over the Schottky uniformization of $\widetilde{C}$ (where $\widetilde{C} \to C$ is the finite covering obtained by extracting an n-th root of the standard multiplicative coordinate U on the Schottky uniformization of C, and $\widetilde{C}$ has "q-parameter" equal to $q^{\frac{1}{n}}$), defines a natural* **restriction morphism**

$$\Xi : (\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}} \to \mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$$

which is **projectively horizontal**. *Here, by "horizontal," we mean with respect to the logarithmic connection (over $S^{\log}$) on the left-hand side defined by Theorem 5.2 and the logarithmic connection (over $S^{\log}$) on the right-hand side defined by the fact that $\mathbf{G}_{\mathrm{m}}$ is defined over $\mathcal{O}$. By "projectively," we mean up to the scalar differential ω_∞ (cf. Theorem 6.1). If one equips the domain $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$ of Ξ with the integral structure obtained by allowing* **"Gaussian poles,"** *then Ξ defines a morphism*

$$\Xi^{\mathrm{GP}} : (\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}} \to \mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$$

which is an **isomorphism over** A*.*

Proof. All of the assertions follow from the above discussion. ○

Remark. In some sense, the comparison isomorphism of Theorem 6.2 may be regarded as the *prototype* of the comparison isomorphism of [Mzk1] (cf. the discussion in the Introduction).

Remark. One superficial difference between the comparison isomorphisms of Theorem 6.2 and [Mzk1] is that unlike the case with the main result of [Mzk1], in the present context, the comparison isomorphism holds *even without twisting* (i.e., even for trivial $\chi_{\mathcal{L}}$, which corresponds to taking the "η" of [Mzk1] to be the origin 0_E). This has to do with the fact that since here we restrict to $\mathbf{G}_{\mathrm{m}}$, rather than to a finite set of torsion points, the case treated here corresponds (from the point of view of [Mzk1]) to the case of twisting by a sort of "generic (hence nontrivial) $\eta \in \mathbf{G}_{\mathrm{m}}$."

Remark. Another way to make an isomorphism out of Ξ (i.e., instead of introducing "Gaussian poles" to form Ξ^{GP}) is to make use of the *theta convolution* — i.e., convolution (relative to Fourier expansions on $\mathbf{G}_{\mathrm{m}}$) with the "theta function" ζ_0^{CG} — cf. the *"theta-convoluted comparison isomorphism"* of [Mzk2], Theorem 10.1. Since the definitions and proofs of the analogue in the present "Schottky-theoretic context" (i.e., as opposed to the "discrete context" of [Mzk1,2]) of [Mzk2], Theorem 10.1, are entirely similar (only technically much simpler!) to those of [Mzk2], we leave their precise formulation (in the present "Schottky-theoretic context") to the reader.

Corollary 6.3. (Description of Monodromy) *The monodromy at* ∞ *of* $((\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}, \nabla_{(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}})$ *(i.e.,* $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$ *equipped with its natural connection) in the logarithmic tangent direction* $\frac{\partial}{\partial \log(q)}$ *is an operator which may be represented as an infinite diagonal matrix whose diagonal entries are given by the* $\Psi(k[r])$*, for* $r \geq 0$*.*

Proof. Clearly, $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$ has *zero monodromy* (since it arises from an object defined over $\mathcal{O}$). Thus, since Ξ^{GP} is an isomorphism, we obtain that $(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}}$ also has zero monodromy. (Note that "projectively" may be ignored as far as monodromy computations are concerned since the differential ω_∞ of Theorem 6.1 has *zero monodromy.*) On the other hand, since the $\widetilde{\zeta}_r^{\mathrm{CG}}$ form a basis of $(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\chi})_{\widehat{S}}$, it thus follows that it suffices to compute the monodromy using the basis $\zeta_r^{\mathrm{CG}} = q^{\Psi(k[r])} \cdot \widetilde{\zeta}_r^{\mathrm{CG}}$ of $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$. Since $\frac{\partial}{\partial \log(q)}(q^{\Psi(k[r])}) = \Psi(k[r]) \cdot q^{\Psi(k[r])}$, Corollary 6.3 follows immediately. ○

Corollary 6.4. (Vanishing of p-Curvature) *The* **p-curvature** *(cf. [Katz1], §5,6, for a discussion of the basic theory of the p-curvature) of* $((\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}, \nabla_{(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}}) \otimes \mathbf{F}_p$ *is identically zero.*

Proof. Since inverting q does not affect the issue of whether or not the p-curvature vanishes identically, it suffices to prove the result for $(\mathcal{V}^{\mathrm{GP}}_{\mathcal{L}^\times})_{\widehat{S}}$. Moreover, since Ξ^{GP} is a projectively horizontal isomorphism, and the differential ω_∞ is the logarithmic exterior derivative of a regular function $\mathfrak{F} \in A^\times$, it suffices to prove the result for $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$. But this is clear, since $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$ admits a horizontal basis (given by the U^k, for $k \in \mathbf{Z}$). ◯

Remark. As remarked earlier, the property of Corollary 6.4 is particularly remarkable in that typically, for $\mathcal{MF}^\nabla$-objects as in [Falt], §2, *zero p-curvature is related to vanishing of the Kodaira-Spencer morphism* (cf. [Katz2]). In the present situation, however, (cf. §8) the corresponding Kodaira-Spencer morphism *does not vanish.* For a substantial generalization of Corollary 6.4, we refer the reader to §7 below.

§7. Crystalline Theta Expansions

In this §, we generalize Corollary 6.4 to show that not only the p-curvature, but also the *"higher p-curvatures"* (cf. [Mzk3], Chapter II, §2.1) of the object considered in Corollary 6.4 *vanish identically.* This property will be used in §8 to complete the proof of Lemma 4.2. Moreover, in the present §, we shall observe that the results that we obtain are sufficient to give a sort of *crystalline analogue of the well-known Fourier expansion* "$\sum q^{\frac{1}{2}k^2} \cdot U^k$" *of a theta function* in a formal neighborhood of the divisor at infinity of the moduli stack of log elliptic curves which is valid in a formal neighborhood of an (essentially) *arbitrary point* (cf. Corollary 7.6 below for details) of this moduli stack (i.e., not just the point at infinity). This construction is of interest in that, by contrast to the *complex case*, where the well-known Fourier expansion "$\sum q^{\frac{1}{2}k^2} \cdot U^k$" is known to be valid not just near infinity, but over the *entire upper half-plane*, up till now (to the knowledge of the author) *no analogue of this expansion has been given* which is valid in a formal neighborhood of an arbitrary point of the moduli stack of elliptic curves.

§7.1. Generalities on Higher p-Curvatures:

In the following, we review what is necessary of the theory of *higher p-curvatures* developed in [Mzk3], Chapter II, §2.1. In particular, it is not necessary (for the purposes of this paper) for the reader to have any knowledge of the theory of [Mzk3].

Let p be a *prime number.* Let A be a *$\mathbf{Z}_p$-flat complete topological ring equipped with the p-adic topology* (i.e., the topology defined by powers of the ideal $p \cdot A$), and $k \stackrel{\mathrm{def}}{=} A \otimes \mathbf{F}_p$. Also, let us suppose that we are given a *p-adic formal scheme S which is formally smooth of relative dimension* 1 *over A.*

In the following discussion, we would like to develop the theory of p^n*-curvatures* of certain types of sheaves equipped with a connection on S. First, let us fix an integer

$n \geq 1$. Let $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ be a *locally free (though not necessarily of finite rank!) quasi-coherent sheaf of* $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$*-modules* (where $S_{\mathbf{Z}/p^n\mathbf{Z}} \stackrel{\text{def}}{=} S \otimes \mathbf{Z}/p^n\mathbf{Z}$). Assume, moreover, that $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is equipped with a *connection* ∇ (relative to the morphism $S \to \mathrm{Spf}(A)$). Then the *p-curvature* of $(\mathcal{E}_{\mathbf{F}_p}, \nabla)$ (where $\mathcal{E}_{\mathbf{F}_p} \stackrel{\text{def}}{=} \mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}} \otimes \mathbf{F}_p$) is a section

$$\mathcal{P}_1 \in \Gamma(S, \Omega^{\mathrm{F}}_{S_{\mathbf{F}_p}} \otimes_{\mathcal{O}_S} \mathrm{End}_{\mathcal{O}_S}(\mathcal{E}_{\mathbf{F}_p}))$$

(cf. [Katz1], §5,6, for a discussion of the basic theory of the p-curvature). Here and in the following discussion, all differentials are *over* A, and we denote the result of base-changing objects on $S_{\mathbf{F}_p}$ via the j-th power of the Frobenius morphism $\Phi_S : S_{\mathbf{F}_p} \to S_{\mathbf{F}_p}$ by means of a superscript F^j. Also, let us write $\mathcal{O}[0] \stackrel{\text{def}}{=} \mathcal{O}_{S_{\mathbf{F}_p}}$, and (for any integer $m \geq 0$)

$$\mathcal{O}[m] \subseteq \mathcal{O}[0]$$

for the *k-subalgebra of $\mathcal{O}[0]$ generated by the image of the p^m-th power map* on $\mathcal{O}[0]$.

One basic property of the p-curvature is the following:

> This section $\mathcal{P}_1 \equiv 0$ if and only if the natural morphism
>
> $$\mathcal{E}^{\nabla}_{\mathbf{F}_p} \otimes_{\mathcal{O}[1]} \mathcal{O}[0] \to \mathcal{E}$$
>
> (where $\mathcal{E}^{\nabla}_{\mathbf{F}_p} \subseteq \mathcal{E}_{\mathbf{F}_p}$ is the $\mathcal{O}[1]$-submodule of horizontal sections) is an *isomorphism.*

Thus, if $\mathcal{P}_1 \equiv 0$, the restriction $\nabla|_{\mathcal{E}_{\mathbf{F}_p}}$ of the connection ∇ to $\mathcal{E}_{\mathbf{F}_p}$ is uniquely determined by the property that it vanishes on $\mathcal{E}^{\nabla}_{\mathbf{F}_p}$.

In particular, the p^j-curvature has been defined for $j = 1$. Let us write $\mathcal{E}[0] \stackrel{\text{def}}{=} \mathcal{E}_{\mathbf{F}_p}$; $\mathcal{E}[1] \stackrel{\text{def}}{=} \mathcal{E}^{\nabla}_{\mathbf{F}_p} \subseteq \mathcal{E}[0]$. Denote the connection induced by ∇ on $\mathcal{E}[0]$ by $\nabla[0]$. Next, assume that:

> *We are given a positive integer $m \leq n$ such that the $p^{m'}$-curvature has been defined and, moreover, vanishes identically for all $m' < m$.*

Suppose, moreover, that under these circumstances:

(1) For all $1 \leq m' < m$, we have defined an $\mathcal{O}[m']$-submodule $\mathcal{E}[m'] \subseteq \mathcal{E}[0]$.

(2) For all $0 \leq m' < m-1$, we have constructed a connection $\nabla[m']$ on the $\mathcal{O}[m']$-module $\mathcal{E}[m']$ whose p-curvature is zero.

(3) For all $1 \leq m' < m$, $\mathcal{E}[m'] = \mathcal{E}[m'-1]^{\nabla[m'-1]}$.

(4) For all $1 \leq m' < m$, $\mathcal{E}[m']$ is generated Zariski locally on $S_{\mathbf{Z}/p^n\mathbf{Z}}$ by sections that lift to sections of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ which are *horizontal modulo* $p^{m'}$.

Observe that:

($*^{\mathrm{LF}}$) Since $\mathcal{O}[0]$ is *finite and flat* over $\mathcal{O}[m']$, and $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is *locally free*, property (3) above implies that, as an $\mathcal{O}[m']$-module, $\mathcal{E}[m']$ enjoys the property that *some direct sum* of a finite number of copies of $\mathcal{E}[m']$ is *locally free.*

Then we would like to construct the p^m*-curvature* of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$, as follows.

First, we would like to construct a *connection* $\nabla[m-1]$ *on* $\mathcal{E}[m-1]$. If $s_{\mathbf{F}_p}$ is a local section of $\mathcal{E}[m-1]$, then let $s_{\mathbf{Z}/p^n\mathbf{Z}}$ be a lifting of $s_{\mathbf{F}_p}$ to $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ which is horizontal modulo p^{m-1} (cf. (4) above). Then $\nabla(s_{\mathbf{Z}/p^n\mathbf{Z}})$ is divisible by p^{m-1}, so it makes sense to define

$$\nabla[m-1](s_{\mathbf{F}_p}) \stackrel{\mathrm{def}}{=} C^{m-1}(p^{-(m-1)} \cdot \nabla(s_{\mathbf{Z}/p^n\mathbf{Z}}) \text{ modulo } p)$$

(where $C^{m-1} : \mathcal{E}[0] \otimes_{\mathcal{O}[0]} \Omega_{S_{\mathbf{F}_p}} \to \mathcal{E}[m-1] \otimes_{\mathcal{O}[m-1]} \Omega_{S_{\mathbf{F}_p}}^{\mathrm{F}^{m-1}}$ is the $(m-1)$-th power of the *Cartier operator* (cf. [Katz1], Theorem 7.2, for basic properties of the Cartier operator); note that this Cartier operator is defined precisely because — cf. (2) above — the p-curvature of $\nabla[m'] \equiv 0$ for $m' < m-1$). It is an easy exercise using the observation ($*^{\mathrm{LF}}$), property (4) above, and Lemma 7.1 below to show that this definition of $\nabla[m-1](s_{\mathbf{F}_p})$ is *independent of the choice of lifting* $s_{\mathbf{Z}/p^n\mathbf{Z}}$. Moreover, it follows immediately from the definition of $\nabla[m-1](s_{\mathbf{F}_p})$ that this correspondence $s_{\mathbf{F}_p} \mapsto \nabla[m-1](s_{\mathbf{F}_p})$ defines a *connection* $\nabla[m-1]$ *on the* $\mathcal{O}[m-1]$*-module* $\mathcal{E}[m-1]$.

We then define the p^m*-curvature of* $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ *to be the* p*-curvature of* $(\mathcal{E}[m-1], \nabla[m-1])$. Moreover, one checks easily (using the observation ($*^{\mathrm{LF}}$), together with Lemma 7.2 below for property (4)) that when this p^m-curvature is identically zero, (if we set $\mathcal{E}[m] \stackrel{\mathrm{def}}{=} \mathcal{E}[m-1]^{\nabla[m-1]}$, then) the above properties (1), ..., (4) are satisfied when "m" is replaced by $m+1$. Thus, by induction, *we obtain a definition of the* p^m*-curvature for all* $m \leq n$.

The following lemmas are well-known:

Lemma 7.1. *Let f be a local section of $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$ which satisfies $df \equiv 0$ modulo p^{n-1}. Suppose that t is a local parameter on S (relative to A). Then f may be written as a power series in terms of the form*

$$c \cdot t^a$$

where $c \in A$, $a \in \mathbf{Z}_{\geq 0}$, and $c \cdot a \equiv 0$ modulo p^{n-1}. In particular, we have $C^n(p^{-(n-1)} \cdot df(\text{modulo } p)) = 0$ *(where C^n is the n-th power of the Cartier operator).*

Proof. The first assertion is clear. Moreover, it implies that $\theta \stackrel{\mathrm{def}}{=} p^{-(n-1)} \cdot df$ (modulo p) is a power series in terms of the form

$$(p^{-(n-1)} \cdot c \cdot a) \cdot t^{a-1}dt$$

But C^n is nonzero only on those terms for which a is *divisible by* p^n; moreover, for such terms, the coefficient $(p^{-(n-1)} \cdot c \cdot a) \equiv 0$ (modulo p). This completes the proof. $\bigcirc$

Lemma 7.2. *Let θ be a local section of the sheaf of differentials $\Omega_{S_{\mathbf{F}_p}}$ which satisfies $C^n(\theta) = 0$. Then there exists a local section f of $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$ such that $df = p^{n-1} \cdot \theta$.*

Proof. Suppose that t is a local parameter of S over A. Let $\mathcal{K} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(C^n)$; write $\mathcal{I}$ for the image modulo p of $p^{-(n-1)} \cdot d$ on those sections of $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$ whose exterior derivative is 0 modulo p^{n-1}. Then both $\mathcal{K}$ and $\mathcal{I}$ form locally free $\mathcal{O}[n]$-submodules of $\mathcal{O}[0] \cdot dt$. It suffices to show that these two submodules coincide. But observe that (by faithfully flat descent) this may be shown after passing to power series in t. Thus, it suffices to show the existence of a *power series* f such that $df = p^{n-1} \cdot \theta$.

Observe that θ may be written as a power series in terms of the form

$$c \cdot t^{a-1} \cdot dt$$

where $c \in A$, $a \in \mathbf{Z}_{\geq 1} \backslash (p^n \cdot \mathbf{Z}_{\geq 1})$. Thus, if we take f to be an appropriate power series in terms of the form

$$(p^{n-1} \cdot a^{-1}) \cdot c \cdot t^a$$

(where we observe that $(p^{n-1} \cdot a^{-1}) \in \mathbf{Z}_p$), we obtain a solution to the equation $df = p^{n-1} \cdot \theta$, as desired. $\bigcirc$

Remark. Note that if, for instance, $1 < m \leq n$, then *it does not make sense to state that "the p^m-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is zero" in the absence of any hypothesis on the p-curvature of* $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$. It does, however, make sense to state that "the p^j-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is zero for all $j \leq m$." That is to say, this statement is to be interpreted as meaning that, first of all, the p-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is zero, which (by the above discussion) allows us to define the p^2-curvature; then that the p^2-curvature is zero, which allows us to define the p^3-curvature, etc. (up to p^m).

We summarize the above discussion as follows:

Theorem 7.3. *Let A be a $\mathbf{Z}_p$-flat complete topological ring equipped with the p-adic topology; $k \stackrel{\mathrm{def}}{=} A \otimes \mathbf{F}_p$; and S be a p-adic formal scheme which is formally smooth of relative dimension 1 over A. Let $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ be a locally free (though not necessarily of finite* *rank!) quasi-coherent sheaf of $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$-modules (for some integer $n \geq 1$), equipped with a connection ∇ over A. Then the p^j-curvature of $(\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}, \nabla)$ is $\equiv 0$ for all $j \leq n$ if and only if $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is generated Zariski locally on $S_{\mathbf{Z}/p^n\mathbf{Z}}$ by horizontal sections.*

Remark. Note, in particular, that the various p^m-curvatures (when they are defined) are *sections of locally free sheaves on $S_{\mathbf{F}_p}$*. Thus, if they vanish over, say, a formal neighborhood of some point of $S_{\mathbf{F}_p}$, then it follows that they vanish over *the schematic closure in $S_{\mathbf{F}_p}$ of that formal neighborhood.*

Corollary 7.4. *Let S and A be as in Theorem 7.3. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules such that $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is locally free (not necessarily of finite rank!) over $S_{\mathbf{Z}/p^n\mathbf{Z}}$ for all integers $n \geq 1$. Suppose, moreover, that $\mathcal{E}$ is equipped with a connection ∇ over A whose p^n-curvature vanishes for all $n \geq 1$. Let $\sigma \in S(A)$ be an A-valued point of S. Write S_σ for the formal completion of S along σ. Then there is a unique isomorphism of $\mathcal{O}_{S_\sigma}$-modules*

$$(\mathcal{E}|_\sigma)\widehat{\otimes}_A\mathcal{O}_{S_\sigma} \cong \mathcal{E}|_{S_\sigma}$$

(where "$\widehat{\otimes}$" denotes the topological tensor product) which (i) is equal to the identity when restricted to σ; and (ii) maps $\mathcal{E}|_\sigma$ on the left-hand side into the set of horizontal sections on the right-hand side.

Proof. Since $\mathcal{E}$ is *topologically* locally free (with respect to the p-adic topology), and the assertion in question is compatible with Zariski localization on $\mathrm{Spec}(A)$, we may assume that $\mathcal{E}|_\sigma$ is a *topologically free* A-module. To prove the existence of a unique isomorphism as asserted, it suffices to show that for any section ϵ of $\mathcal{E}|_\sigma$, there exists a unique horizontal section $\widetilde{\epsilon}$ of $\mathcal{E}|_{S_\sigma}$ lifting ϵ. Write $\mathcal{I}$ for the sheaf of ideals defining the subobject σ inside S_σ. By Theorem 7.3, it follows that for any $n \geq 1$, there exists a section ϵ_n of S_σ which lifts ϵ and whose reduction modulo p^n is horizontal. Moreover, by the "description of 'f' as a power series" given in Lemma 7.1, it follows that as $n \to \infty$, the ϵ_n converge in the $(\mathcal{I}^p, p)$-adic topology to a unique horizontal section ϵ of $\mathcal{E}|_{S_\sigma}$, as desired. ◯

Corollary 7.5. *Let A be a $\mathbf{Z}$-flat commutative ring, and S a smooth A-scheme of relative dimension 1 over A. Let $\mathcal{E}$ be a quasi-coherent sheaf of locally free (not necessarily of finite rank!) $\mathcal{O}_S$-modules. Suppose, moreover, that $\mathcal{E}$ is equipped with a connection ∇ over A such that for every prime number p, the p^n-curvature of $(\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}, \nabla_{\mathbf{Z}/p^n\mathbf{Z}})$ vanishes for all $n \geq 1$. Let $\sigma \in S(A)$ be an A-valued point of S. Write S_σ for the formal completion of S along σ. Then there is a unique isomorphism of $\mathcal{O}_{S_\sigma}$-modules*

$$(\mathcal{E}|_\sigma)\widehat{\otimes}_A\mathcal{O}_{S_\sigma} \cong \mathcal{E}|_{S_\sigma}$$

(where "$\widehat{\otimes}$" denotes the topological tensor product) which (i) is equal to the identity when restricted to σ; and (ii) maps $\mathcal{E}|_\sigma$ on the left-hand side into the set of horizontal sections on the right-hand side.

Proof. Write S_σ^{PD} for the (completed) divided power envelope of σ inside S. Thus, if t is a local parameter for S over A at σ (i.e., such that $t = 0$ on σ), then $\mathcal{O}_{S_\sigma^{\mathrm{PD}}}$ consists of power series of the form

$$\sum_{n \geq 0} c_n \cdot \frac{t^n}{n!}$$

(where $c_n \in A$). In particular, it follows from the elementary theory of integrable connections that we get a unique isomorphism

$$(\mathcal{E}|_\sigma)\widehat{\otimes}_A \mathcal{O}_{S_\sigma^{\mathrm{PD}}} \cong \mathcal{E}|_{S_\sigma^{\mathrm{PD}}}$$

satisfying (i), (ii). To check that this isomorphism is, in fact, defined over $\mathcal{O}_{S_\sigma}$ (as asserted in Corollary 7.5) — i.e., to check that the denominators do not, in fact, appear — it suffices to see that this holds after S and A are replaced by their p-adic completions. But then the assertion to be checked is precisely the content of Corollary 7.4. ○

§7.2. Application to Theta Expansions:

Now we apply the theory of the above discussion to the objects considered in Corollary 6.4.

Corollary 7.6. **(Crystalline Theta Expansions)** *Let A be a* **Z***-flat commutative ring, and S a smooth A-scheme of relative dimension* 1 *over A. Let $f : E \to S$ be an* **elliptic curve** *over S, equipped with an* **ample line bundle** *$\mathcal{L}$ of the form $\mathcal{O}_E(\eta)$, where $\eta \in E(S)$ is a* **torsion point**. *Also, let us assume that for every prime number p, the generic member of this family of elliptic curves in each irreducible component of $S_{\mathbf{F}_p}$ is* **ordinary** *(i.e., has nonzero Hasse invariant), and that the condition $(*^{\mathrm{KS}})$ of Lemma 5.1 is satisfied for $T \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)$ (equipped with the trivial log structure). Set $* \stackrel{\mathrm{def}}{=} \mathcal{L}$, and write $\nabla_{\mathcal{V}_\mathcal{L}}$ for the connection on*

$$\mathcal{V}_\mathcal{L} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{\mathrm{et}}})$$

*determined by Theorem 5.2 for some choice of a rigidification of $\mathcal{L}$ at the origin 0_E of E. Then $\mathcal{V}_\mathcal{L}$ is locally free on S, and the p^n-***curvature** *of the pair $(\mathcal{V}_\mathcal{L}, \nabla_{\mathcal{V}_\mathcal{L}})$ is $\equiv 0$ for all integers $n \geq 1$.*

Let $\sigma \in S(A)$ be an A-valued point of S. Write S_σ for the formal completion of S along σ. Then there is a unique isomorphism of $\mathcal{O}_{S_\sigma}$-modules

$$(\mathcal{V}_\mathcal{L}|_\sigma)\widehat{\otimes}_A \mathcal{O}_{S_\sigma} \cong \mathcal{V}_\mathcal{L}|_{S_\sigma}$$

(where "$\widehat{\otimes}$" denotes the topological tensor product) which (i) is equal to the identity when restricted to σ; and (ii) maps $\mathcal{V}_{\mathcal{L}}|_\sigma$ on the left-hand side into the set of horizontal sections on the right-hand side.

In particular, the expansion of a section of $F^1(\mathcal{V}_{\mathcal{L}})|_{S_\sigma} = f_*(\mathcal{L})|_{S_\sigma}$ as a topological $\mathcal{O}_{S_\sigma}$-linear combination of elements of some basis of $\mathcal{V}_{\mathcal{L}}|_\sigma$ may be regarded as a sort of "crystalline analogue" of the usual expansion "$\sum q^{\frac{1}{2}k^2} \cdot U^k$" of a theta function (cf. the Remark below for more details).

Remark. It is natural to regard the expansion discussed at the end of the statement of Corollary 7.6 as a "crystalline analogue" of the usual expansion "$\sum q^{\frac{1}{2}k^2} \cdot U^k$" of a theta function for the following reason: The usual expansion is precisely an expansion in terms of the U^k (for $k \in \mathbf{Z}$), which, as we saw in §6, form (via the isomorphism Ξ^{GP}) a *horizontal* (topological) *basis* of $\mathcal{V}_{\mathcal{L}}$ in a formal neighborhood of infinity.

Proof. (of Corollary 7.6) The proof of the vanishing of the higher p-curvatures is entirely the same as that of Corollary 6.4 (cf. also the Remark following Theorem 7.3) — i.e., one uses the projectively horizontal isomorphism Ξ^{GP} of Theorem 6.2, together with the fact that $\mathcal{V}_{\mathbf{G}_{\mathrm{m}}}$ admits a *horizontal basis*, given by the U^k, for $k \in \mathbf{Z}$ (cf. Theorem 7.3). Note that here we use the assumption that the family of elliptic curves in question is generically ordinary in characteristic p in order to reduce (cf. the Remark following Theorem 7.3) to the situation studied in §6 in a formal neighborhood of infinity. The remaining assertions are formal consequences of Corollary 7.5. ○

Remark. Although in Corollary 7.6, we took $\mathcal{L}$ to be of degree 1, it is not difficult to derive from Corollary 7.6 (using the theory of theta groups — cf. the discussion of §8.1) the corresponding result for $\mathcal{L}$ of arbitrary degree. We leave the routine details to the reader.

§8. "Griffiths Semi-Transversality"

In this §, we investigate the relationship between the *connections* defined in §5 and the *Hodge filtrations* (determined by torsorial degree) on various spaces of functions on the universal extension. In some sense, the central phenomenon here is the fact that unlike many well-known objects with similar structures for which the connection gives rise to *jumps of length* 1 in the Hodge filtration (i.e., "Griffiths transversality"), in the present situation, the connection gives rise to *jumps of length* 2 in the Hodge filtration. We refer to this phenomenon as *"Griffiths semi-transversality."* At a more concrete level, this phenomenon may be thought of as the *quadratic* (i.e., nonlinear) nature of the exponent of the q-parameter in the well-known theta expansion "$\sum q^{\frac{1}{2}k^2} \cdot U^k$." Moreover, it turns out that this phenomenon gives rise to interesting (non-trivial) *Kodaira-Spencer morphisms*, as well as *Hasse-type invariants*. In the present §, in addition to calculating these objects, we apply these calculations at the prime $p = 2$ to complete the proof of Lemma 4.2 of §4.4.

§8.1. The Kodaira-Spencer Morphism of the Crystalline Theta Object:

In the following discussion, we use the notation of §5.2. Moreover, we let $T \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbf{Z})$ (equipped with the trivial log structure), so S is *smooth over* $\mathbf{Z}$, and we assume that the log structure on S is defined by a $\mathbf{Z}$-*flat divisor with normal crossings* $\subseteq S$, and that the condition $(*^{\mathrm{KS}})$ of Lemma 5.1 is satisfied. (For instance, a typical example of such an $S^{\log}$ is given by any étale morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}[\frac{1}{2}]}$ to the moduli stack of log elliptic curves over $\mathbf{Z}[\frac{1}{2}]$.)

Next, let us assume that we are given a *degree one ample line bundle* $\mathcal{L}$ *of torsion type* (cf. Definition 4.4) on C (i.e., the semi-stable compactification of E). Note that

$$\mathbf{R}^1 f_*(\mathcal{L}) = 0$$

(where, in the following discussion, we shall (by abuse of notation) denote all structure morphisms to S by "f"). Indeed, this follows from Serre duality, and the fact that $f_*(\omega_E \otimes \mathcal{L}^{-1}) = 0$, since $\mathcal{L}$ is ample. Also, let us assume that we are given a *rigidification*

$$\mathcal{L}_\epsilon \cong \mathcal{O}_S$$

(where $\mathcal{L}_\epsilon$ is the restriction of $\mathcal{L}$ to the zero section of $C \to S$) at the zero section of $C \to S$.

Set $* \stackrel{\mathrm{def}}{=} \mathcal{L}$. By Theorem 5.2, we thus obtain a *logarithmic connection* on the pair

$$(E^*_{C,\mathrm{et}}, \mathcal{L}_{E^*_{C,\mathrm{et}}})$$

(where $\mathcal{L}_{E^*_{C,\mathrm{et}}} \stackrel{\mathrm{def}}{=} \mathcal{L}|_{E^*_{C,\mathrm{et}}}$), regarded as a family of polarized varieties over S. Since taking global sections is a natural operation, we thus get an *induced logarithmic connection* $\nabla_{\mathcal{V}_\mathcal{L}}$ on the quasi-coherent sheaf of $\mathcal{O}_S$-modules given by

$$\mathcal{V}_\mathcal{L} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{C,\mathrm{et}}})$$

Note that in addition to the connection $\nabla_{\mathcal{V}_\mathcal{L}}$, the sheaf $\mathcal{V}_\mathcal{L}$ is also equipped with a *"Hodge filtration"* $F^r(\mathcal{V}_\mathcal{L})$ induced by the filtration on $\mathcal{R}^{\mathcal{L},\mathrm{et}}$ (cf. Theorem 4.3) with subquotients given by

$$(F^{r+1}/F^r)(\mathcal{V}_\mathcal{L}) = \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$$

(where we use the fact that $\mathbf{R}^1 f_*(\mathcal{L}) = 0$).

Remark. Note that the data $(\mathcal{V}_{\mathcal{L}}, F^r(\mathcal{V}_{\mathcal{L}}), \nabla_{\mathcal{V}_{\mathcal{L}}})$ is reminiscent of the *"$\mathcal{MF}^{\nabla}$-objects"* of [Falt], §2. There are, of course, various obvious differences: e.g., $\mathcal{V}_{\mathcal{L}}$ is "a vector bundle of *infinite rank*"; we have yet to define a *Frobenius action* (though the author hopes to address this issue in a future paper). One more subtle difference is the *behavior of the connection $\nabla_{\mathcal{V}_{\mathcal{L}}}$ relative to the Hodge filtration* — cf. the discussion of "Griffiths semi-transversality" below. Yet another more subtle difference is the relationship between the *p-curvature* (cf. the Remark following Corollary 6.4) and the *Kodaira-Spencer morphism* (to be studied in detail below). Nevertheless, *the analogy with the $\mathcal{MF}^{\nabla}$-objects of [Falt], §2, is one of the fundamental motivations for the present paper.* It is the hope of the author to develop this point of view further in future papers.

Note that if $\mathcal{L}$ and its rigidification are *symmetric* (i.e., preserved by the natural action of ± 1 on C), then ± 1 *acts naturally on* $(\mathcal{V}_{\mathcal{L}}, F^r(\mathcal{V}_{\mathcal{L}}), \nabla_{\mathcal{V}_{\mathcal{L}}})$. If, moreover, 2 is *invertible on* S, then considering eigenspaces for this action gives rise to a *natural splitting*

$$\mathcal{V}_{\mathcal{L}} = \mathcal{V}_{\mathcal{L}}{}^{+} \oplus \mathcal{V}_{\mathcal{L}}{}^{-}$$

which is compatible with $F^r(\mathcal{V}_{\mathcal{L}})$ and $\nabla_{\mathcal{V}_{\mathcal{L}}}$. In particular, since the relative degree of $\mathcal{L}$ over S is one, ± 1 acts *trivially* on $F^1(\mathcal{V}_{\mathcal{L}}) = f_*(\mathcal{L})$, while -1 acts as -1 on τ_E. Thus, in this case, ± 1 acts as $(\pm 1)^r$ on

$$(F^{r+1}/F^r)(\mathcal{V}_{\mathcal{L}}) = \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$$

i.e., for $\ddagger \in \{+, -\}$,

$$(F^{r+1}/F^r)(\mathcal{V}_{\mathcal{L}}{}^{\ddagger}) = (F^{r+1}/F^r)(\mathcal{V}_{\mathcal{L}})$$

if the sign of $(-1)^r$ is equal to $\ddagger$, and $(F^{r+1}/F^r)(\mathcal{V}_{\mathcal{L}}{}^{\ddagger}) = 0$ otherwise.

Next, we would like to consider the *extent to which the connection $\nabla_{\mathcal{V}_{\mathcal{L}}}$ preserves the Hodge filtration.* First, let us recall the isomorphism

$$\Xi_{E_C^{\dagger}} : \pi_1^*(E_C^{\dagger}) \cong \pi_2^*(E_C^{\dagger})$$

of the discussion preceding Lemma 5.1. Note that this isomorphism is well-known to preserve the Hodge filtration on $\mathcal{O}_{E_C^{\dagger}}$ *up to a jump in the index which is* ≤ 1, i.e., $F^r(-)$ on one side is not necessarily sent into $F^r(-)$ on the other, but it *is* sent into F^{r+1}. This property of "connections giving rise to jumps of magnitude ≤ 1" is often referred to as

Griffiths transversality (cf., e.g., [Falt], §2). In the case of the universal extension, this Griffiths transversality follows easily from the definition of the universal extension as the moduli space of certain line bundles with connection (cf. [Mzk1], Chapter III, §1).

Note that since $E_C^\dagger$ and $E^*_{C,\mathrm{et}}$ are the *same* in characteristic zero, it follows that

$$\Xi_{E^*_{C,\mathrm{et}}} : \pi_1^*(E^*_{C,\mathrm{et}}) \cong \pi_2^*(E^*_{C,\mathrm{et}})$$

(cf. Theorem 5.2) also *satisfies Griffiths transversality.*

Now let us consider the connection on the pair $(E^*_{C,\mathrm{et}}, \mathcal{L}_{E^*_{C,\mathrm{et}}})$. That is to say, we would like to compare $\pi_1^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ with $\pi_2^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ via $\Xi_{E^*_{C,\mathrm{et}}}$, over the base $S^{\log} \times^{\mathrm{PD},2} S^{\log}$. Write $\mathcal{I}$ for the ideal defining the diagonal in $S^{\log} \times^{\mathrm{PD},2} S^{\log}$. Then let us observe that *both $\pi_1^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ and $\pi_2^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ define line bundles on $\pi_2^*(E^*_{C,\mathrm{et}})$ which agree modulo $\mathcal{I}$ and which are defined by transition functions that belong to $F^2(-)$ of the structure sheaf of $\pi_2^*(E^*_{C,\mathrm{et}})$*. Indeed, the transition functions defining $\pi_2^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ belong to $F^1(-) \subseteq F^2(-)$ since this line bundle is pulled back from the line bundle $\pi_2^*\mathcal{L}$ on π_2^*C. On the other hand, $\pi_1^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$, regarded as a line bundle on $\pi_1^*(E^*_{C,\mathrm{et}})$, is (by the same reasoning) defined by transition functions that belong to $F^1(-)$ of the structure sheaf of $\pi_1^*(E^*_{C,\mathrm{et}})$, but these transition functions are mapped into $F^2(-)$ of the structure sheaf of $\pi_2^*(E^*_{C,\mathrm{et}})$ (by the Griffiths transversality of $\Xi_{E^*_{C,\mathrm{et}}}$).

Thus, in summary, in order to obtain an (or — equivalently, by Theorem 4.3, (III) — the unique) isomorphism between $\pi_1^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$ and $\pi_2^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$, it suffices to introduce $F^r(-)$ for a sufficiently large r such that the *deformation class* that gives the difference between these two line bundles *vanishes*. Since (by the discussion of the preceding paragraph) this deformation class lies in $F^2(-)$, we conclude, by Theorem 4.3, (II), that it suffices to pass to $F^3(-)$, in order to cause this deformation to vanish. Thus, in summary, we started with an object (i.e., $\pi_1^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$) which lies in $F^1(-)$ on $\pi_1^*(E^*_{C,\mathrm{et}})$, then transported this object via $\Xi_{E^*_{C,\mathrm{et}}}$ to an object lying in $F^2(-)$ on $\pi_2^*(E^*_{C,\mathrm{et}})$ which, when regarded as an object lying in $F^3(-) = F^{1+2}(-)$ (on $\pi_2^*(E^*_{C,\mathrm{et}})$), is isomorphic to $\pi_2^*\mathcal{L}_{E^*_{C,\mathrm{et}}}$. That is to say, we have shown the following:

> *The connection on $(E^*_{C,\mathrm{et}}, \mathcal{L}_{E^*_{C,\mathrm{et}}})$ preserves the Hodge filtration up to jumps of magnitude ≤ 2.*

We will refer to this property of preserving the Hodge filtration up to jumps of magnitude ≤ 2 as **Griffiths semi-transversality** (by analogy to "Griffiths transversality," in the case of jumps of magnitude ≤ 1).

Next, we would like to consider the *"Kodaira-Spencer morphism"* obtained from $(\mathcal{V}_\mathcal{L}, F^r(\mathcal{V}_\mathcal{L}), \nabla_{\mathcal{V}_\mathcal{L}})$ by looking at the extent to which $F^1(\mathcal{V}_\mathcal{L})$ is preserved by $\nabla_{\mathcal{V}_\mathcal{L}}$. More precisely, we would like to consider the composite:

$$f_*(\mathcal{L}) = F^1(\mathcal{V}_{\mathcal{L}}) \quad \hookrightarrow \quad \mathcal{V}_{\mathcal{L}} \quad \stackrel{\nabla_{\mathcal{V}_{\mathcal{L}}}}{\longrightarrow} \quad \mathcal{V}_{\mathcal{L}} \otimes_{\mathcal{O}_S} \Omega_{S^{\log}} \quad \rightarrow \quad (\mathcal{V}_{\mathcal{L}}/F^2(\mathcal{V}_{\mathcal{L}})) \otimes_{\mathcal{O}_S} \Omega_{S^{\log}}$$

By "Griffiths semi-transversality," we see that the above composite maps into

$$\frac{1}{2} \cdot \tau_E^{\otimes 2} \otimes_{\mathcal{O}_S} f_*(\mathcal{L}) \otimes_{\mathcal{O}_S} \Omega_{S^{\log}} = (F^3/F^2)(\mathcal{V}_{\mathcal{L}}) \otimes_{\mathcal{O}_S} \Omega_{S^{\log}} \subseteq (\mathcal{V}_{\mathcal{L}}/F^2(\mathcal{V}_{\mathcal{L}})) \otimes_{\mathcal{O}_S} \Omega_{S^{\log}}$$

Moreover, the above composite is easily seen to be $\mathcal{O}_S$-*linear*, hence (by taking duals) defines a morphism

$$\kappa_{\mathcal{L}} : \Theta_{S^{\log}} \rightarrow End(f_*(\mathcal{L})) \otimes_{\mathcal{O}_S} \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

(where $\Theta_{S^{\log}}$ is the dual of $\Omega_{S^{\log}}$, and "*End*" denotes the sheaf of $\mathcal{O}_S$-endomorphisms of the locally free $\mathcal{O}_S$-module $f_*(\mathcal{L})$).

Now, since in the present discussion, $f_*(\mathcal{L})$ forms a *line bundle* on S, it follows that: $End(f_*(\mathcal{L}) = \mathcal{O}_S$. Thus, $\kappa_{\mathcal{L}}$ is a morphism

$$\kappa_{\mathcal{L}} : \Theta_{S^{\log}} \rightarrow \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

By considering the *universal case* (i.e., when $S = (\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, except with differentials "$\frac{1}{2} \cdot \Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}}$" in order to take account of $(*^{\mathrm{KS}})$), we thus obtain that $\kappa_{\mathcal{L}}$ (for arbitrary $C^{\log} \rightarrow S^{\log}$) arises from some element

$$\in \Gamma((\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}, \frac{1}{2} \cdot \Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}} \otimes \frac{1}{2} \cdot \tau_E^{\otimes 2}) = \frac{1}{4} \cdot \mathbf{Z}$$

(since, as is well-known, the Kodaira-Spencer morphism of the first de Rham cohomology module of the tautological log elliptic curve over $(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}$ defines an isomorphism $\Omega_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}} \otimes \frac{1}{2} \cdot \tau_E^{\otimes 2} \cong \frac{1}{2} \cdot \mathcal{O}_{(\overline{\mathcal{M}}^{\log}_{1,0})_{\mathbf{Z}}}$). That is to say, $\kappa_{\mathcal{L}}$ *is equal to some universal constant* $\in \frac{1}{4} \cdot \mathbf{Z}$. Now I *claim* that

$(*^{\mathrm{UC}})$ $\kappa_{\mathcal{L}} = \frac{1}{2}$.

In this paper, we will present *two independent proofs* of the fact that $\kappa_{\mathcal{L}} \neq 0$. In the *first proof*, we prove the stronger result $(*^{\mathrm{UC}})$. In the *second proof*, we show only that $\kappa_{\mathcal{L}} \neq 0$. Thus, *logically speaking, the second proof is unnecessary.* The reasons for the inclusion of the second proof in this paper are that:

(i) The author finds this proof to be aesthetically appealing.

(ii) This proof shows how one can "almost prove" $(*^{\mathrm{UC}})$ using only elementary algebraic geometry (whereas the first proof requires the "more advanced" techniques of §6).

First Proof that $\kappa_{\mathcal{L}} \neq 0$:

In this proof, we use the notation of §6. In the context of §6, the Kodaira-Spencer morphism in question arises from applying the connection on $(\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}}$ to $F^1((\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}})$ and looking at the image of this map in $(F^3/F^2)((\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}})$. By Theorem 6.1 (and the fact that scalar differentials have no effect on the Kodaira-Spencer morphism), we thus see that this amounts to *differentiating* $\Xi(\zeta_0^{\mathrm{CG}})$ (which generates $F^1((\mathcal{V}^*_{\mathcal{L}^\chi})_{\widehat{S}})$) in, say, the logarithmic tangent direction $\frac{\partial}{\partial \log(q)}$, noting that this results in a linear combination of $\Xi(\zeta_0^{\mathrm{CG}}), \Xi(\zeta_1^{\mathrm{CG}}), \Xi(\zeta_2^{\mathrm{CG}})$, and, finally, observing that the coefficient of $\Xi(\zeta_2^{\mathrm{CG}})$ is what we would expect (relative to all the canonical identifications involved) if the Kodaira-Spencer morphism is to equal "$\frac{1}{2}$," as desired.

Since

$$\Xi(\zeta_0^{\mathrm{CG}}) = \sum_{k \in \mathbf{Z}} \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{1}{2}k^2 + \frac{i_\chi}{n}k} \cdot U^k$$

we thus see that applying $\frac{\partial}{\partial \log(q)}$ amounts to multiplying the k-th coefficient in the series by $\frac{1}{2}k^2 + \frac{i_\chi}{n}k$, hence *yields the same series as that obtained by applying* $P(\frac{\partial}{\partial \log(U)})$, where $P(X) \in \mathbf{Q}[X]$ is a polynomial with rational coefficients of degree 2 whose *leading term is* $\frac{1}{2}X^2$. Relative to all the canonical identifications involved, this leading term means precisely that the "Kodaira-Spencer morphism" in question is $\frac{1}{2}$, as desired (at least in a neighborhood of infinity; but this is sufficient, since (cf. the discussion above) the Kodaira-Spencer morphism is a "constant"). This completes the proof of $(*^{\mathrm{UC}})$. ○

Remark. Finally, we remark that the above proof, which involves the ideas surrounding the comparison isomorphism of Theorem 6.2, further justifies the assertion of the author in [Mzk1] that *the "arithmetic Kodaira-Spencer morphism" constructed in [Mzk1], Chapter IX, is indeed analogous to the usual geometric Kodaira-Spencer morphism* (cf. the discussion in the Introduction).

We summarize the above discussion as follows:

Theorem 8.1. (The Crystalline Theta Object) *Let S be a smooth $\mathbf{Z}$-scheme of finite type, equipped with a log structure defined by a $\mathbf{Z}$-flat divisor with crossings $\subseteq S$. Let*

$C^{\log} \to S^{\log}$ be a **log elliptic curve**, *equipped with a degree one ample line bundle $\mathcal{L}$ of torsion type (cf. Definition 4.4) on C, and a rigidification $\mathcal{L}_\epsilon \cong \mathcal{O}_S$ of this line bundle over the zero section of $C^{\log}$. Also, we assume that the condition $(*^{\mathrm{KS}})$ of Lemma 5.1 is satisfied.*

Then this data defines a natural logarithmic connection $\nabla_{\mathcal{V}_\mathcal{L}}$ on the quasi-coherent $\mathcal{O}_S$-module

$$\mathcal{V}_\mathcal{L} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{C,\mathrm{et}}})$$

This $\mathcal{O}_S$-module $\mathcal{V}_\mathcal{L}$ is equipped with a natural Hodge filtration $F^r(\mathcal{V}_\mathcal{L})$, whose subquotients are given by

$$(F^{r+1}/F^r)(\mathcal{V}_\mathcal{L}) = \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$$

Moreover, the triple $(\mathcal{V}_\mathcal{L}, F^r(\mathcal{V}_\mathcal{L}), \nabla_{\mathcal{V}_\mathcal{L}})$ is compatible with automorphisms of the given data. (We leave it to the reader to write out the routine details.) In particular, if $\mathcal{L}$ and its rigidification are **symmetric** *(i.e., preserved by the natural action of ± 1 on $C^{\log}$), and $2 \in \mathcal{O}_S^\times$, then considering the eigenspaces of this action gives rise to a natural direct sum decomposition*

$$\mathcal{V}_\mathcal{L} = {\mathcal{V}_\mathcal{L}}^+ \oplus {\mathcal{V}_\mathcal{L}}^-$$

such that (for $\ddagger \in \{+,-\}$) $(F^{r+1}/F^r)({\mathcal{V}_\mathcal{L}}^\ddagger)$ is $= (F^{r+1}/F^r)(\mathcal{V}_\mathcal{L})$ if the sign of $(-1)^r$ is equal to $\ddagger$, and $= 0$ otherwise.

The natural logarithmic connections on $(E^*_{C,\mathrm{et}}, \mathcal{L}_{E^*_{C,\mathrm{et}}})$ (cf. Theorem 5.2) and $\mathcal{V}_\mathcal{L}$ satisfy **"Griffiths semi-transversality"** *with respect to the natural Hodge filtrations (i.e., the connections preserve the Hodge filtrations up to jumps of magnitude ≤ 2). Thus, $\nabla_{\mathcal{V}_\mathcal{L}}$ induces a* **Kodaira-Spencer morphism**

$$\kappa_\mathcal{L} : \Theta_{S^{\log}} \to \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

which (modulo the identification $\tau_E^{\otimes 2} \cong \Theta_{\overline{\mathcal{M}}_{1,0}}|_S$) is equal to $\frac{1}{2}$ times the **usual Kodaira-Spencer morphism**.

Proof. Everything follows from the above discussion. ○

Remark. In anticipation of the definition of some sort of natural Frobenius action on $\mathcal{V}_\mathcal{L}$ (cf. the Remark on $\mathcal{MF}^\nabla$-objects at the beginning of this §), we shall refer to the triple

$$(\mathcal{V}_\mathcal{L}, F^r(\mathcal{V}_\mathcal{L}), \nabla_{\mathcal{V}_\mathcal{L}})$$

as the *crystalline theta object*.

Before presenting the "Second Proof" referred to above, we would like to pause to review (those aspects that will be necessary in the present paper of) the *theory of theta* *groups*. This theory will be of use not only in the "Second Proof," but also in various other arguments that we use in the remainder of this paper.

Let $l \geq 2$ be an *integer*, and assume that we are given a *family of elliptic curves*

$$E \to S$$

over a base scheme S which satisfies the hypotheses of Theorem 8.1 (with trivial log structure), and on which (for simplicity) l is *invertible*. Also, we assume that we are given a *degree one line bundle of torsion type* $\mathcal{L}$ as in Theorem 8.1. Let us write

$$\widetilde{E} \to E$$

for the (étale) isogeny given by multiplication by l on $\widetilde{E} \stackrel{\text{def}}{=} E$. It is worth noting here that since $\widetilde{\mathcal{L}} \stackrel{\text{def}}{=} \mathcal{L}|_{\widetilde{E}}$ will then be of degree $l^2 \geq 4 \geq 3$, it follows that $\widetilde{\mathcal{L}}$ is *very ample* (over S) and, moreover, (for $n \geq 1$)

$$\mathbf{S}^n\{f_*(\widetilde{\mathcal{L}})\} \to f_*(\widetilde{\mathcal{L}}^{\otimes n})$$

is *surjective* (cf., e.g., [Mumf4], §2, Theorem 6). *It is these properties of very ampleness and surjectivity that often make it desirable to use the line bundle* $\widetilde{\mathcal{L}}$*, rather than the (simpler and more readily analyzable) degree one line bundle* $\mathcal{L}$*.*

Now let us write (for an S-scheme T)

$$\mathcal{G}_{\widetilde{\mathcal{L}}}(T) \stackrel{\text{def}}{=} \{(\alpha, \iota) \mid \alpha \in \widetilde{E}(T),\ \iota : \mathcal{T}_\alpha^* \widetilde{\mathcal{L}}_T \cong \widetilde{\mathcal{L}}_T\}$$

for the *theta group* associated to $\widetilde{\mathcal{L}}$ (cf. [Mumf1,2,3]; [Mumf5], §23; or, alternatively, [Mzk1], Chapter IV, §1, for an exposition of the theory of theta groups). (Here, we write $\mathcal{T}_\alpha : \widetilde{E}_T \to \widetilde{E}_T$ for the automorphism given by translation by α.) Thus, $\mathcal{G}_{\widetilde{\mathcal{L}}}$ fits into an exact sequence:

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}_{\widetilde{\mathcal{L}}} \to {}_{l^2}E \to 1$$

(where ${}_{l^2}E$ is the kernel of multiplication by l^2 on E).

Now observe that since l is invertible on S, $\widetilde{E} \to E$ induces a natural morphism $\widetilde{E}^\dagger \to E^\dagger$ (cf. Theorem 5.3) which, in turn, induces an isomorphism $\widetilde{E}^\dagger \cong E^\dagger \times_E \widetilde{E}$, as well as a morphism $\widetilde{E}^\dagger_{\text{et}} \to E^\dagger_{\text{et}}$ (cf. Theorem 5.3). Moreover, the kernel of $\widetilde{E}^\dagger_{\text{et}} \to E^\dagger_{\text{et}}$ projects isomorphically into ${}_l\widetilde{E} \subseteq \widetilde{E}$, i.e., we have an exact sequence

$$0 \to {}_l\widetilde{E} \to \widetilde{E}^\dagger_{\text{et}} \to E^\dagger_{\text{et}} \to 0$$

of group schemes over S. Next, observe that it follows from the fact that $\widetilde{\mathcal{L}}$ is obtained by pull-back from E that we have a natural subgroup scheme

$$H \subseteq \mathcal{G}_{\widetilde{\mathcal{L}}} \times_{\widetilde{E}} \widetilde{E}^{\dagger}_{\mathrm{et}}$$

(describing how to descend $\widetilde{\mathcal{L}}$ back down to $\mathcal{L}$) that maps isomorphically to ${}_l\widetilde{E}$. Thus, H acts naturally on

$$\mathcal{V}_{\widetilde{\mathcal{L}}} \stackrel{\mathrm{def}}{=} f_*(\widetilde{\mathcal{L}}|_{E^*_{\mathrm{et}} \times_E \widetilde{E}})$$

(where $* \stackrel{\mathrm{def}}{=} \mathcal{L}$). Moreover, the isogeny $\widetilde{E} \to E$ induces an inclusion $\mathcal{V}_{\mathcal{L}} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{\mathrm{et}}}) \hookrightarrow \mathcal{V}_{\widetilde{\mathcal{L}}}$ which is *compatible with connections and Hodge filtrations* and which *identifies* $\mathcal{V}_{\mathcal{L}}$ *with the* H*-invariants* $\mathcal{V}_{\widetilde{\mathcal{L}}}^H$ *of* $\mathcal{V}_{\widetilde{\mathcal{L}}}$. Conversely, the theory of theta groups assures us that $\mathcal{V}_{\widetilde{\mathcal{L}}}$ may be recovered from the data $(\mathcal{V}_{\mathcal{L}}, H)$. Thus, in summary,

> *Although it is frequently necessary to work with* $\widetilde{\mathcal{L}}$*, and, in particular, with* $\mathcal{V}_{\widetilde{\mathcal{L}}}$ *in order to make use of the very ampleness of* $\widetilde{\mathcal{L}}$*, the theory of theta groups tells us that the structure of* $\mathcal{V}_{\widetilde{\mathcal{L}}}$ *may be analyzed (at least in principle) by using the "simpler" object* $\mathcal{V}_{\mathcal{L}}$*.*

We are now ready to present the "Second Proof" of the nonvanishing of $\kappa_{\mathcal{L}}$.

Second Proof that $\kappa_{\mathcal{L}} \neq 0$*:*

In this proof, we work over an S which is étale over $(\mathcal{M}_{1,0})_{\mathbf{Q}}$. Thus, in particular, *the various integral structures considered on* $E^{\dagger}$ *all coincide.* Also, we use the notation $\widetilde{E} \to E$, $\widetilde{\mathcal{L}}$, $\mathcal{V}_{\widetilde{\mathcal{L}}}$, of the review of theta groups given above.

Now observe that just as we defined a Kodaira-Spencer morphism for $\mathcal{V}_{\mathcal{L}}$, we may define an analogous morphism

$$\kappa_{\widetilde{\mathcal{L}}} : \Theta_S \to \mathit{End}(f_*(\widetilde{\mathcal{L}})) \otimes_{\mathcal{O}_S} \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

for $\mathcal{V}_{\widetilde{\mathcal{L}}}$. Note that since the action of H on $\mathcal{V}_{\widetilde{\mathcal{L}}}$ is clearly *horizontal*, it follows that $\kappa_{\widetilde{\mathcal{L}}}$ factors through

$$\mathit{End}_H(f_*(\widetilde{\mathcal{L}})) \otimes_{\mathcal{O}_S} \frac{1}{2} \cdot \tau_E^{\otimes 2} \subseteq \mathit{End}(f_*(\widetilde{\mathcal{L}})) \otimes_{\mathcal{O}_S} \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

where we use the subscript "H" to denote endomorphisms that commute with the action of H. Since $\mathit{End}_H(f_*(\widetilde{\mathcal{L}})) = \mathit{End}(f_*(\mathcal{L})) = \mathcal{O}_S$, we thus obtain a morphism

$$\Theta_S \to \frac{1}{2} \cdot \tau_E^{\otimes 2}$$

which (by the evident *horizontality* of the natural inclusion $\mathcal{V}_{\mathcal{L}} \hookrightarrow \mathcal{V}_{\widetilde{\mathcal{L}}}$) may be *identified with* $\kappa_{\mathcal{L}}$. In particular, if $\kappa_{\mathcal{L}} = 0$, then it follows that $\kappa_{\widetilde{\mathcal{L}}} = 0$, as well. Put another way, this means that, $\nabla_{\mathcal{V}_{\mathcal{L}}}$, $\nabla_{\mathcal{V}_{\widetilde{\mathcal{L}}}}$ *map* $F^1(\mathcal{V}_{\mathcal{L}})$, $F^1(\mathcal{V}_{\widetilde{\mathcal{L}}})$ *into* $F^2(\mathcal{V}_{\mathcal{L}})$, $F^2(\mathcal{V}_{\widetilde{\mathcal{L}}})$, respectively.

Next, let us observe that since ± 1 acts *horizontally* (i.e., in such a way that $\nabla_{\mathcal{V}_{\mathcal{L}}}$ is preserved) on $\mathcal{V}_{\mathcal{L}}$, and $\deg(\mathcal{L}) = 1$, it follows from the above discussion that $\nabla_{\mathcal{V}_{\mathcal{L}}}$ *always maps* $F^1(\mathcal{V}_{\mathcal{L}}) \subseteq {\mathcal{V}_{\mathcal{L}}}^+$ *into* ${\mathcal{V}_{\mathcal{L}}}^+$. Since ${\mathcal{V}_{\mathcal{L}}}^+ \bigcap F^2(\mathcal{V}_{\mathcal{L}}) \subseteq F^1(\mathcal{V}_{\mathcal{L}})$, we thus obtain that $\nabla_{\mathcal{V}_{\mathcal{L}}}$ *preserves* $F^1(\mathcal{V}_{\mathcal{L}})$. By considering the action of H as above, we thus conclude that $\nabla_{\mathcal{V}_{\widetilde{\mathcal{L}}}}$ *preserves* $F^1(\mathcal{V}_{\widetilde{\mathcal{L}}})$. In particular, we thus obtain a connection ∇_1 on

$$\mathcal{F}_1 \stackrel{\mathrm{def}}{=} f_*(\widetilde{\mathcal{L}}) = F^1(\mathcal{V}_{\widetilde{\mathcal{L}}})$$

Moreover, since the natural morphism

$$\mathbf{S}^n(\mathcal{F}_1) \to \mathcal{F}_n \stackrel{\mathrm{def}}{=} f_*(\widetilde{\mathcal{L}}^{\otimes n}) \subseteq \mathcal{V}_{\widetilde{\mathcal{L}}^{\otimes n}} \stackrel{\mathrm{def}}{=} f_*(\widetilde{\mathcal{L}}^{\otimes n}|_{\widetilde{E}^\dagger})$$

(for $n \geq 1$) given by multiplication of sections is clearly *horizontal*, and the morphism $\mathbf{S}^n(\mathcal{F}_1) \to \mathcal{F}_n$ is *surjective* (cf. the review of theta groups given above), we thus obtain that the *entire graded* $\mathcal{O}_S$*-algebra*

$$\bigoplus_{n \geq 0} \mathcal{F}_n$$

(where we let $\mathcal{F}_0 \stackrel{\mathrm{def}}{=} \mathcal{O}_S$) *admits a connection.* Since "*Proj*" of this graded algebra *is* the elliptic curve $\widetilde{E} \to S$, we thus obtain that the family of elliptic curves $\widetilde{E} \cong E \to S$ *admits a connection.* But (unraveling the definitions) this implies that the classifying morphism $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Q}}$ for this family $E \to S$ has *Kodaira-Spencer morphism* (i.e., derivative) equal to 0, which contradicts the fact that we took $S \to (\overline{\mathcal{M}}_{1,0})_{\mathbf{Q}}$ to be *étale.* This completes the "Second Proof" that $\kappa_{\mathcal{L}} \neq 0$. ◯

Before proceeding, we note that the technique of the above proof (involving theta groups and taking "Proj") motivates the following immediate consequence of Corollary 7.6:

Corollary 8.2. (Higher p-Curvatures of the Polarized Universal Extension) *In the notation of Corollary 7.6, all of the higher p-curvatures of the pair* $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ **vanish** *identically. Moreover, if* σ, S_σ *are as in Corollary 7.6, then there is a unique isomorphism*

$$\{(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})|_\sigma\} \widehat{\otimes}_A \mathcal{O}_{S_\sigma} \cong (E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})|_{S_\sigma}$$

over S_σ (where "$\widehat{\otimes}$" denotes the topological tensor product) which (i) is equal to the identity when restricted to σ; and (ii) maps $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})|_\sigma$ on the left-hand side into the set of horizontal sections on the right-hand side.

Remark. We leave it to the reader to formulate the routine details of defining the "higher p-curvatures of the nonlinear object $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$."

Finally, before proceeding, we note one other important consequence of the above discussion. Recall the condition $(*^{\mathrm{KS}})$ of Lemma 5.1. Because we used this condition in the proof of Lemma 5.1, it became necessary to assume this condition whenever we wished to make use of a *connection on* $E^*_{C,\mathrm{et}}$ (where $*$ is as in Lemma 5.1) — cf. Theorems 5.2, 5.3, 8.1; Corollaries 7.6, 8.2. The above discussion shows, however, that in fact, *this condition is not necessary.* Indeed, by translating as in the proof of Lemma 5.1, it suffices to verify this in the case where $*$ is the line bundle defined by the origin. Moreover, by working in the universal case, it suffices to show that *arbitrary* tangent vectors on $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}_2}$ (i.e., not just tangent vectors divisible by 2) act in an *integral* fashion on $E^*_{C,\mathrm{et}}$ via the connection of Theorem 5.2. But observe that this may be verified in a neighborhood of infinity. Moreover, by working with *theta groups* and the isogeny $\widetilde{E} \stackrel{\mathrm{def}}{=} E \to E$ given by multiplication by an odd integer $l \geq 1$ (cf. the review of theta groups given above), we see that it suffices to check the asserted integrality for the action of the connection on $\mathcal{V}_{\mathcal{L}} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^*_{C,\mathrm{et}}})$, where $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(0_E)$. But this then amounts (cf. the "First Proof" above) to the assertion that $\frac{\partial}{\partial \log(q)}$ acts integrally on $\mathcal{V}_{\mathcal{L}}$, which is generated by the "ζ^{CG}_r," i.e., by series of the form

$$\sum_{k \in \mathbf{Z}} P(k) \cdot \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{1}{2}(k^2-k)} \cdot U^k$$

where $P(-)$ is an *integer-valued polynomial with rational coefficients* (cf. the explicit description of $\Xi(\zeta^{\mathrm{CG}}_r)$ given in §6). (Note that here, since "$\chi_{\mathcal{L}}$" is trivial, $\frac{i_\chi}{n} = -\frac{1}{2}$.) But acting on such a series by $\frac{\partial}{\partial \log(q)}$ amounts to replacing $P(T)$ (where T is an indeterminate) by $\frac{1}{2}(T^2 - T) \cdot P(T)$, which is *still an integer-valued* polynomial with rational coefficients. Thus, $\frac{\partial}{\partial \log(q)}$ acts *integrally* on $\mathcal{V}_{\mathcal{L}}$, as desired. In summary:

Corollary 8.3. *The assumption that the condition $(*^{\mathrm{KS}})$ be satisfied in Theorems 5.2, 5.3, 8.1; Corollaries 7.6, 8.2; may be omitted without affecting the validity of these results.*

§8.2. An Explicit Formula for the Higher p-Curvatures:

In this §, we give an *explicit formula for calculating the higher p-curvatures* defined in §7.1. This formula is obtained in the context of the general discussion of §7.1, and has nothing to do with the "Hodge-Arakelov theory of elliptic curves." In §8.3 below, we apply this formula to obtain a certain "higher p-curvature version" of the result concerning the Kodaira-Spencer morphism given in Theorem 8.1.

In the following discussion, we use the notation of §7.1. In particular, we fix a *prime number* p, and let A be a $\mathbf{Z}_p$*-flat complete topological ring equipped with the* p*-adic topology*, and $k \stackrel{\text{def}}{=} A \otimes \mathbf{F}_p$. Also, let us suppose that we are given a p*-adic formal scheme* S *which is formally smooth of relative dimension* 1 *over* A, together with a *locally free (though not necessarily of finite rank!) quasi-coherent sheaf* $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ *of* $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$*-modules* (where $n \geq 1$ is an integer, and $S_{\mathbf{Z}/p^n\mathbf{Z}} \stackrel{\text{def}}{=} S \otimes \mathbf{Z}/p^n\mathbf{Z}$) equipped with a *connection* $\nabla_{\mathcal{E}}$ (relative to the morphism $S \to \mathrm{Spf}(A)$). In addition, we assume that we are given a *local coordinate* t (of S over A), and that we wish to compute the higher p-curvatures of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ in a *"punctured formal neighborhood"* $\mathrm{Spec}(A[[t]][t^{-1}])$ of $V(t) \subseteq S$. In the discussion to follow, all of our derivatives will be in the direction $d\ \log(t) \stackrel{\text{def}}{=} \frac{dt}{t}$. Thus, for instance, we will write ∇ (respectively, D) for the result of applying $\nabla_{\mathcal{E}}$ (respectively, the exterior derivative d) in the tangent direction defined by (the dual of) $d\ \log(t)$.

Often in our discussion, we shall wish to consider various *differential operators* acting on $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$. Many of these differential operators will be constructed from simpler differential operators by substituting the simpler differential operators into various formal polynomials. The most fundamental formal polynomial in the following discussion is given by:

$$\Psi(T) \stackrel{\text{def}}{=} \binom{T}{p} = \frac{1}{p!} T(T-1) \cdot \ldots \cdot (T-(p-1)) \in \mathbf{Q}_p[T]$$

Also, we shall write

$$\Psi^{\{j\}}(T) \stackrel{\text{def}}{=} \Psi(\Psi(\ldots \Psi(T)))$$

for the result of iterating Ψ a total of j times (where $j \geq 1$ is an integer).

Lemma 8.4. *Suppose that the* p^j*-curvature of* $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ *vanishes identically for all* $j \leq n$. *Then*

$$(p \cdot \Psi^{\{j\}})(\nabla) \equiv 0 \text{ modulo } p$$

for all $j \leq n$.

Proof. Since the p^j-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ vanishes identically for all $j \leq n$, it follows that $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is locally generated by horizontal sections (cf. Theorem 7.3). Thus, it suffices to prove Lemma 8.4 in the case where $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}} = \mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$ equipped with the trivial connection. In particular, if suffices to show that for any series

$$f = \sum_{k>>-\infty} c_k \cdot t^k$$

(where $c_k \in A$), the coefficients of each of the

$$\Psi^{\{j\}}(D) \cdot f$$

are $\in A$ (i.e., are integral). But this amounts to showing the integrality of the

$$\Psi^{\{j\}}(k)$$

for $k \in \mathbf{Z}$, which follows from the well-known fact that $\Psi(-)$ maps integers to integers. $\bigcirc$

Now suppose that $n \geq 2$, and that the p^j-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ *vanishes identically* for all $j \leq n-1$. Then it follows that $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is locally generated by sections that are horizontal modulo p^{n-1}. Thus, for an appropriate local basis, ∇ may be written in the form:

$$\nabla = D + p^{n-1} \cdot \omega$$

where ω is a section of $End(\mathcal{E}_{\mathbf{F}_p})$. Let us compute $(p \cdot \Psi)(\nabla)$. Since $(p^{n-1} \cdot \omega)^2 = 0$, it follows that

$$\begin{aligned}
(p \cdot \Psi)(\nabla) = (p \cdot \Psi)(D) &+ \{(p-1)!\}^{-1} \cdot p^{n-1} \cdot \\
&\sum_{j=0}^{p-1} D \cdot \ldots \cdot (D - (j-1)) \cdot \omega \cdot (D - (j+1)) \cdot \ldots \cdot (D - (p-1)) \\
&= (p \cdot \Psi)(D) + p^{n-1} \cdot \mathrm{Lin}_\omega\{(p \cdot \Psi)(D + \omega)\} \\
&= (p \cdot \Psi)(D) - p^{n-1} \cdot \mathrm{Lin}_\omega\{(D + \omega)^p - (D + \omega)\} \\
&= (p \cdot \Psi)(D) - p^{n-1} \cdot \{D^{p-1}(\omega) - \omega\}
\end{aligned}$$

where "$\mathrm{Lin}_\omega(-)$" denotes the "term of degree one in ω"; $(p-1)! \equiv -1$ modulo p (by an easy calculation using the fact that $\mathbf{F}_p{}^\times$ is *cyclic*); and the equation

$$\mathrm{Lin}_\omega\{(D + \omega)^p\} = D^{p-1}(\omega)$$

follows from *Jacobson's formula* (see, e.g., [Jac], pp. 186-187). This formula states that if a and b are elements of an associative ring R of characteristic p, then

$$(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)$$

where the $s_i(a,b)$ are given by the formula:

$$(\mathrm{ad}(Ta+b))^{p-1}(a) = \sum_{i=1}^{p-1} i s_i(a,b) T^{i-1}$$

computed in the ring $R[T]$, where T is an indeterminate. (Here, we apply this formula in the case $a \stackrel{\mathrm{def}}{=} \omega$; $b \stackrel{\mathrm{def}}{=} D$, and use the fact that if $f \in \mathcal{O}_{S_{\mathbf{F}_p}}$, then $(\mathrm{ad}(b))(f) = [b,f] = [D,f] = D(f)$.) On the other hand, an easy computation reveals that:

$$-(D^{p-1}(\omega) - \omega) = C(\omega)$$

(i.e., the result of applying the *Cartier operator* to ω). (Indeed, the point here is that (in characteristic p) $D^{p-1}(t^j) - t^j = (j^{p-1} - 1) \cdot t^j$, which is $= -t^j$ if j is divisible by p, and $= 0$ otherwise.)

Thus, in summary, we obtain that modulo p^{n-1}:

$$\Psi(\nabla) \equiv D_1 + p^{n-2} \cdot \omega_1$$

where $D_1 \stackrel{\mathrm{def}}{=} \Psi(D)$, and $\omega_1 \stackrel{\mathrm{def}}{=} C(\omega)$.

Next, let us observe that the operator D_1 acts as $\frac{1}{p} \cdot D$ on t^{pj} (where $j \in \mathbf{Z}$) modulo p. Indeed, this follows from the fact that

$$\binom{pj}{p} \equiv j \cdot (p-1)! \cdot \{(p-1)!\}^{-1} \equiv j$$

(modulo p). Since ω_1 may be written as a series in terms of the form t^{pj}, it thus follows that *we may repeat the above calculation* with ∇ replaced by $\Psi(\nabla)$. In particular, if we repeat this calculation over and over again, we see that we obtain the following result:

Lemma 8.5. *For $1 \leq j \leq n-1$, we have: modulo p^{n-j},*

$$\Psi^{\{j\}}(\nabla) \equiv D_j + p^{n-1-j} \cdot \omega_j$$

where $D_j \stackrel{\mathrm{def}}{=} \Psi^{\{j\}}(D)$, and $\omega_j \stackrel{\mathrm{def}}{=} C^j(\omega)$.

In particular, observe that if we take $j = n-1$, then the right-hand side of the equality in Lemma 8.5 — regarded as an operator on series in terms of the form $t^{p^{n-1} \cdot a}$ (where $a \in \mathbf{Z}$) in characteristic p — is simply the connection denoted "$\nabla[n-1]$" in §7.1. (Here, we use that $D_{n-1}(t^{p^{n-1} \cdot a}) \equiv a \cdot t^{p^{n-1} \cdot a}$, which follows from Lemma 8.7 below.) Thus, in particular, if we substitute

$$\Psi^{\{n-1\}}(\nabla) \equiv D_{n-1} + \omega_{n-1}$$

into the polynomial $(p \cdot \Psi)(T) \equiv -(T^p - T)$ (modulo p), we obtain an operator

$$-\{(D_{n-1} + \omega_{n-1})^p - (D_{n-1} + \omega_{n-1})\}$$

which, when restricted to series in terms of the form $t^{p^{n-1} \cdot a}$ in characteristic p, is equal to *minus the p^n-curvature $\mathcal{P}_n$ of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$*. On the other hand, by Jacobson's formula (quoted above), it follows that this operator differs from $-(D_{n-1}^p - D_{n-1})$ by an operator which is *$\mathcal{O}_S$-linear*. Moreover, the operator $-(D_{n-1}^p - D_{n-1})$ is easily seen to be $\equiv 0$ modulo p (cf. Lemma 8.4 above). Thus, in summary, we see that we have proven the following: modulo p,

$$(p \cdot \Psi^{\{n\}})(\nabla) = -\mathcal{P}_n$$

(where both sides are linear over $\mathcal{O}_S$). We state this as a theorem:

Theorem 8.6. (Explicit Formula for Higher p-Curvatures) *Let $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ be a quasi-coherent sheaf of locally free $\mathcal{O}_{S_{\mathbf{Z}/p^n\mathbf{Z}}}$-modules (where $n \geq 1$ is an integer, and $S_{\mathbf{Z}/p^n\mathbf{Z}} \stackrel{\text{def}}{=} S \otimes \mathbf{Z}/p^n\mathbf{Z}$) equipped with a connection $\nabla_{\mathcal{E}}$ (relative to the morphism $S \to \mathrm{Spf}(A)$). Suppose that we are also given a local coordinate t (of S over A). Write ∇ for the result of applying $\nabla_{\mathcal{E}}$ (respectively, the exterior derivative d) in the tangent direction defined by (the dual of) $d\log(t) \stackrel{\text{def}}{=} \frac{dt}{t}$. Also, let us write*

$$\Psi(T) \stackrel{\text{def}}{=} \binom{T}{p} = \frac{1}{p!} T(T-1) \cdot \ldots \cdot (T-(p-1)) \in \mathbf{Q}_p[T]$$

(where T is an indeterminate) and

$$\Psi^{\{j\}}(T) \stackrel{\text{def}}{=} \Psi(\Psi(\ldots \Psi(T)))$$

for the result of iterating Ψ a total of j times (where $j \geq 1$ is an integer). Suppose that the p^j-curvature of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is $\equiv 0$ for all $j \leq n-1$. Then the p^n-curvature $\mathcal{P}_n$ of $\mathcal{E}_{\mathbf{Z}/p^n\mathbf{Z}}$ is given by the formula:

$$\mathcal{P}_n \equiv -(p \cdot \Psi^{\{n\}})(\nabla)$$

(modulo p).

Remark. In fact, it is not difficult to see that instead of using the specific polynomial $\Psi(T)$, we could have used *any* polynomial $F(T) \in \frac{1}{p} \cdot \mathbf{Z}_p[T]$ of degree p such that $p \cdot F(T) \equiv p \cdot \Psi(T)$ (modulo p). Similarly, if one is interested in *differentiating in the tangent direction dual to*

dt (i.e., rather that $d\log(t)$), then a similar formula to that of Theorem 8.6 holds, except with $\Psi(T)$ replaced by the polynomial $\frac{1}{p!}T^p$ (or, indeed, any polynomial $F(T) \in \frac{1}{p}\mathbf{Z}_p[T]$ of degree p such that $p \cdot F(T) \equiv p \cdot (\frac{1}{p!}T^p)$ (modulo p)). Since these variants may be proved by exactly the same argument as that given above, and, in the present paper, we only need the explicit formula stated in Theorem 8.6, we leave the precise statement and proof of these variants as (easy!) exercises for the reader.

Lemma 8.7. *Modulo p, we have:*

$$\Psi^{\{n-1\}}(p^{n-1} \cdot a) \equiv a$$

(for $a \in \mathbf{Z}$).

Proof. Indeed, this follows from recursive application of the formula:

$$\Psi(p^m \cdot a) \equiv p^{m-1} \cdot a \cdot (p-1)! \cdot \{(p-1)!\}^{-1} \equiv p^{m-1} \cdot a$$

modulo p^m (where $m \geq 1$ is an integer). ○

§8.3. Hasse-type Invariants of the Crystalline Theta Object:

In this §, we study the relationship between the *higher p-curvatures* and the *Hodge filtration* of the crystalline theta object. In the case of the first de Rham cohomology module of an elliptic curve in positive characteristic, the invariant that describes the relationship between the p-curvature and the Hodge filtration is referred to as the "Hasse invariant." Thus, one may think of the invariants discussed in the present § as *analogues of the Hasse invariant for the crystalline theta object*. In addition to computing these invariants, we apply our computations to compute the proof of Lemma 4.2 of §4.4.

In this §, we let p be a *prime number*. In the following discussion, we shall use the notation of §4.4. Thus, we let S be étale over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$, and $E \to S$ be the pull-back from $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ of the *tautological elliptic curve* over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$. Also, let us write $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(0_E)$, and $S^{\mathrm{ord}} \subseteq S$ for the open subscheme obtained by *removing the supersingular points* in characteristic p.

Recall the *intermediate étale integral structures* $E_{\mathrm{et}}^{*;\{N\}}$ (where $* \stackrel{\mathrm{def}}{=} \mathcal{L}$) defined at the end of §4.4. Note that in the present *p-adic context*, we have:

$$E_{\mathrm{et}}^{*;\{N\}} = E_{\mathrm{et}}^{*;\{p^j\}}$$

where $j \geq 0$ is the largest integer such that $p^j \leq N$. (Indeed, this can be seen, for instance, by using the description at the end of §4.4 of the subquotients $(F^{r+1}/F^r)(-)$

of the intermediate étale integral structures, together with the elementary numerical fact that if $p^j \leq N < p^{j+1}$, then

$$\mathrm{ord}_p(N!) = \sum_{i=0}^{j} a_i \cdot \mathrm{ord}_p\{(p^i)!\}$$

(where $N = \sum_{i=0}^{j} a_i p^i$; $a_i \in \{0, 1, 2, \ldots, p-1\}$).) Now we have the following observation:

Lemma 8.8. *The connection of Theorem 5.2 (cf. also Corollary 8.3) on* $(E^*_{\mathrm{et}}, \mathcal{L}_{E^*_{\mathrm{et}}})$ *in fact acts integrally on* $(E^{*;\{N\}}_{\mathrm{et}}, \mathcal{L}_{E^{*;\{N\}}_{\mathrm{et}}})$ *for all integers* $N \geq 2$.

Proof. Indeed, the argument is that of the discussion preceding Corollary 8.3: Namely, the point is that the exponent of q in the expansion

$$\sum_{k \in \mathbf{Z}} P(k) \cdot \chi_{\mathcal{M}}(k_{\mathrm{et}}) \cdot q^{\frac{1}{2}(k^2-k)} \cdot U^k$$

is given by $\frac{1}{2}k(k-1)$, which is equal to $\binom{k}{2}$. Thus, multiplication by this factor preserves all intermediate integral structures (for which $N \geq 2$). $\bigcirc$

Now let us write

$$\mathcal{V}_{\mathcal{L}}^{\{N\}} \stackrel{\mathrm{def}}{=} f_*(\mathcal{L}_{E^{*;\{N\}}_{\mathrm{et}}})$$

Thus, $\mathcal{V}_{\mathcal{L}}^{\{N\}}$ admits a natural *connection* $\nabla_{\mathcal{V}}$ (as soon as we choose a rigidification of $\mathcal{L}_\epsilon$ — cf. Theorem 8.1). Now let $r \geq 0$; $a, b \geq 1$ be *integers*. Suppose that the *p^j-curvature of $\mathcal{V}_{\mathcal{L}}^{\{p^b\}}$ vanishes* for $j < a$. Then if we compose the inclusion $F^{r+1}(\mathcal{V}_{\mathcal{L}}^{\{p^b\}}) \subseteq \mathcal{V}_{\mathcal{L}}^{\{p^b\}}$ with the *p^a-curvature* of $\mathcal{V}_{\mathcal{L}}^{\{p^b\}}$, we obtain a morphism

$$F^{r+1}(\mathcal{V}_{\mathcal{L}}^{\{p^b\}})_{\mathbf{F}_p} \otimes_{\mathcal{O}_S} \tau_E^{\otimes\ 2p^a} \to (\mathcal{V}_{\mathcal{L}}^{\{p^b\}})_{\mathbf{F}_p}$$

(where we note that (by our assumption that $S \to (\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ is *étale*) we may identify $\tau_E^{\otimes\ 2}$ with the tangent bundle of S over $\mathbf{Z}_p$). Note that by the *explicit formula for the p^a-curvature* (Theorem 8.6), it follows that the p^a-curvature may be computed as a polynomial in $\nabla_{\mathcal{V}}$ of degree $\leq p^a$. Since — by *Griffiths semi-transversality* (cf. Theorem 8.1) — $\nabla_{\mathcal{V}}$ gives rise to jumps of length ≤ 2 in the Hodge filtration, it thus follows that the above morphism maps into $F^{r+1+2p^a}(\mathcal{V}_{\mathcal{L}}^{\{p^b\}})_{\mathbf{F}_p} \subseteq (\mathcal{V}_{\mathcal{L}}^{\{p^b\}})_{\mathbf{F}_p}$ (and, similarly, maps $F^r(-)$ into $F^{r+2p^a}(-)$). Thus, by projecting onto $(F^{r+1+2p^a}/F^{r+2p^a})(-)$, we obtain a morphism

$$\left\{\frac{1}{\gamma(p^b,r)}\cdot f_*(\mathcal{L})\otimes_{\mathcal{O}_S}\tau_E^{\otimes\ r+2p^a}\right\}_{\mathbf{F}_p}\to\left\{\frac{1}{\gamma(p^b,r+2p^a)}\cdot f_*(\mathcal{L})\otimes_{\mathcal{O}_S}\tau_E^{\otimes\ r+2p^a}\right\}_{\mathbf{F}_p}$$

(where $\gamma(-,-)$ is as in the discussion at the end of §4.4), i.e., (since $f_*(\mathcal{L})$ and τ_E are line bundles on S) an element

$$h_{p;r;a,b}\in\left\{\frac{\gamma(p^b,r)}{\gamma(p^b,r+2p^a)}\cdot\mathcal{O}_S\right\}\otimes\mathbf{F}_p$$

which may be regarded as a sort of generalization of the *Hasse invariant* of an elliptic curve in that it describes the extent to which the (higher) p-curvature is compatible with the Hodge filtration.

Theorem 8.9. (Computation of Hasse-type Invariants of the Crystalline Theta Object) *Let p be a* **prime number**. *Suppose that S is* **étale** *over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$, and write $E\to S$ for the pull-back from $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$ of the* **tautological elliptic curve** *over $(\mathcal{M}_{1,0})_{\mathbf{Z}_p}$. Also, let us write $\mathcal{L}\stackrel{\mathrm{def}}{=}\mathcal{O}_E(0_E)$. Let $r\geq 0$; $a,b\geq 1$ be integers, and suppose that the p^j-curvature of $\mathcal{V}_{\mathcal{L}}^{\{p^b\}}$* **vanishes** *for $j<a$. Then the invariant*

$$h_{p;r;a,b}\in\left\{\frac{\gamma(p^b,r)}{\gamma(p^b,r+2p^a)}\cdot\mathcal{O}_S\right\}\otimes\mathbf{F}_p$$

(where $\gamma(-,-)$ is the number defined at the end of §4.4) obtained by restricting the p^a-curvature to

$$F^{r+1}(\mathcal{V}_{\mathcal{L}}^{\{p^b\}})\subseteq\mathcal{V}_{\mathcal{L}}^{\{p^b\}}\stackrel{\mathrm{def}}{=}f_*(\mathcal{L}_{E_{\mathrm{et}}^{*;\{p^b\}}})$$

and then projecting onto $(F^{r+1+2p^a}/F^{r+2p^a})(-)$ is equal to the image of

$$\left(\frac{1}{2}\right)^{p^a}\cdot\left\{\frac{-p}{(p!)^{p^{a-1}}\cdot(p!)^{p^{a-2}}\cdot\ldots\cdot(p!)^p\cdot p!}\right\}\in\left(\frac{1}{2}\right)^{p^a}\cdot\left\{\frac{p}{(p^a)!}\right\}\cdot\mathbf{Z}_p^{\times}$$

in $\{\frac{\gamma(p^b,r)}{\gamma(p^b,r+2p^a)}\cdot\mathcal{O}_S\}\otimes\mathbf{F}_p$.

Proof. Indeed, it suffices to compute $h_{p;a,b}$ near infinity using Theorem 8.6. Then $h_{p;r;a,b}$ is simply the leading term of $-p\cdot\Psi^{\{a\}}(\frac{1}{2}k(k-1))$. Each iteration of $\Psi(-)$ affects the leading term by raising it to the p-th power, and then dividing it by $p!$. Thus, induction on a gives rise to the expression stated in Theorem 8.9. ○

Remark. Thus, Theorem 8.9 may be interpreted as a sort of "higher p-curvature" generalization of the portion of Theorem 8.1 concerning the Kodaira-Spencer morphism of the crystalline theta object (which, so to speak, corresponds to the case "$a = 0$"). Note that one may also think of Theorem 8.9 as a result that *reduces the computation of the various (arithmetic-geometric!) Hasse-type invariants of the crystalline theta object to a matter of combinatorics.*

One might suspect from the appearance of the number 2 throughout the statement of Theorem 8.9 that the content of this theorem takes on a *particularly interesting form* when one restricts to the case $p = 2$. This is indeed the case. In the following discussion, we would like to make this intuition explicit. Thus, for the remainder of this §, *we assume that* $p = 2$. First, let us observe that the *leading term of*

$$P_a(T) \stackrel{\text{def}}{=} \Psi^{\{a\}}(\frac{1}{2}T(T-1))$$

is (up to a $\mathbf{Z}_p{}^{\times}$-multiple) of the form

$$\frac{1}{(p^{a+1})!} \cdot T^{p^{a+1}}$$

(since, in the present situation, $2^{p^a} \cdot (p^a)! \in (p^{a+1})! \cdot \mathbf{Z}_p{}^{\times}$). Thus, in particular, it follows that for any integer $a \geq 0$, the $\mathbf{Z}_p$-subalgebra of $\mathbf{Q}_p[T]$ generated by $P_0(T) = \frac{1}{2}T(T-1), P_1(T), \ldots, P_a(T)$ coincides with the $\mathbf{Z}_p$-subalgebra generated by $\{\binom{T}{r}\}$, where r ranges over all powers of p which are $\leq p^{a+1}$. On the other hand, since vanishing of the p^a-curvature is equivalent to the *integrality* of $\Psi^{\{a\}}(\nabla_{\mathcal{V}})$ (cf. Theorem 8.6), we thus obtain the following result:

Corollary 8.10. (Higher p-Curvatures in the Case p=2) *In the context of Theorem 8.9, assume that $p = 2$. Then the p^a-curvature of $\mathcal{V}_{\mathcal{L}}^{\{p^b\}}$* **vanishes** *for all $a < b$. Moreover, the p^a-curvature of $\mathcal{V}_{\mathcal{L}}^{\{p^a\}}$ maps $F^1(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) \otimes \tau_E^{\otimes\ p^{a+1}} \otimes \mathbf{F}_p$ isomorphically onto the image of the morphism*

$$p \cdot F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}}) \subseteq F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) \to F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})_{\mathbf{F}_p}$$

Finally, the composite of the p^a-curvature of $\mathcal{V}_{\mathcal{L}}^{\{p^a\}}$ with the natural projection to the subquotient $(F^{1+p^{a+1}}/F^{p^{a+1}})(-)$ maps $F^1(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) \otimes \tau_E^{\otimes\ p^{a+1}} \otimes \mathbf{F}_p$ isomorphically onto the reduction of

$$(F^{1+p^{a+1}}/F^{p^{a+1}})(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) = p \cdot (F^{1+p^{a+1}}/F^{p^{a+1}})(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}})$$

modulo p.

Proof. All of the assertions follow from the above discussion, together with the fact that $\{(p^a)!\}^p \cdot p \in (p^{a+1})! \cdot \mathbf{Z}_p{}^{\times}$. ○

We are now ready to apply Theorem 8.9 in the case of $p = 2$ to complete the proof of Lemma 4.2:

Completion of the Proof of Lemma 4.2:

By applying the *theory of theta groups* (as reviewed in §8.1— cf. especially the "very ampleness" and "surjectivity" properties that were discussed there), we reduce immediately to the case (cf. the notation of Lemma 4.2) where the line bundle "$\mathcal{M}$" is equal to $\mathcal{L}$. Then we must show the surjectivity of

$$(F^{r+1}/F^r)(\mathcal{V}_{\mathcal{L}}) \to \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$$

for all $r \geq 0$. We propose to do this by *induction on* r. Note that this surjectivity is clear for $r = 0, 1$. Moreover, let us observe that for $r = 2$, this surjectivity follows from the description of the *Kodaira-Spencer morphism* in Theorem 8.1. Indeed, this description implies that if we take apply $\nabla_{\mathcal{V}}$ (in a generating tangent direction at some point of S) to a generator of $F^1(\mathcal{V}_{\mathcal{L}})$, then the image in $(F^3/F^2)(\mathcal{V}_{\mathcal{L}})$ of the resulting section of $F^3(\mathcal{V}_{\mathcal{L}})$ generates $\frac{1}{2} \cdot \tau_E^{\otimes 2} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$, as desired. This completes the proof of this surjectivity for $r \leq 2$. It turns out that the proof for $r > 2$ will proceed in a similar (although somewhat more intricate) fashion by applying Corollary 8.10 in place of Theorem 8.1.

Next, let us observe that once we show this surjectivity for all $r \leq N$ (where $N \geq 1$ is some integer), it follows from the theory of theta groups (as discussed above) that the corresponding surjectivity for $r \leq N$ and "$\mathcal{M}$" equal to sufficiently large tensor powers of $\mathcal{L}$ *also holds.* In particular, by the definition of the "intermediate étale integral structures" (i.e., as consisting of those functions lying in the algebra generated by $F^{r+1}(-)$ of the "full" étale integral structure) we obtain (by taking "Proj") that *the subquotients of the structure sheaf of* $E_{\mathrm{et}}^{*;\{N\}}$ *have the expected form* (as given at the end of §4.4). Thus, in summary, we see that if we show the desired surjectivity for all $r \leq p^a$ (where $a \geq 1$ is an integer), then this already *implies* the desired surjectivity for all $r < p^{a+1}$.

In particular, we see that *it suffices to prove the desired surjectivity for* $r = p^{a+1}$ *(where* $a \geq 1$ *is an integer), under the assumption that the subquotients of the structure sheaf of* $E_{\mathrm{et}}^{*;\{p^a\}}$ *have the expected form* (as given at the end of §4.4). But now let us observe that

$$F^j(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) = F^j(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}})$$

for $j \leq p^{a+1}$, while (cf. Corollary 8.10)

$$(F^{1+p^{a+1}}/F^{p^{a+1}})(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})|_{S^{\mathrm{ord}}} = p \cdot (F^{1+p^{a+1}}/F^{p^{a+1}})(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}})|_{S^{\mathrm{ord}}}$$

(i.e., we restrict to the ordinary locus S^{ord} since over S^{ord} the desired surjectivity is already known). Thus, in particular, the desired surjectivity for $r = p^{a+1}$ will follow as soon as we show that the torsion-free sheaf $(F^{1+p^{a+1}}/F^{p^{a+1}})(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}})$ is a *line bundle* over all of S. But this will follow as soon as we show that the *splitting* σ_a of the morphism

$$F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})_{\mathbf{F}_p} \to \left\{\frac{p}{(p^{a+1})!} \cdot \tau_E^{\otimes\ p^{a+1}} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})\right\}_{\mathbf{F}_p}$$

defined over $S^{\mathrm{ord}}_{\mathbf{F}_p}$ by taking the image of the morphism

$$p \cdot F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^{a+1}\}}) \subseteq F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}}) \to F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})_{\mathbf{F}_p}$$

extends over $S_{\mathbf{F}_p}$.

On the other hand, by (the portion concerning $\mathcal{V}_{\mathcal{L}}^{\{p^a\}}$ of) Corollary 8.10 (which, by the induction hypothesis, we are free to use), if we take a generator of $F^1(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})_{\mathbf{F}_p}$ near an *arbitrary* point of $S_{\mathbf{F}_p}$ and apply to this generator the p^a*-curvature* of $\mathcal{V}_{\mathcal{L}}^{\{p^a\}}$ (in a generating tangent direction), the image of the resulting section of $F^{1+p^{a+1}}(\mathcal{V}_{\mathcal{L}}^{\{p^a\}})_{\mathbf{F}_p}$ in $\{\frac{p}{(p^{a+1})!} \cdot \tau_E^{\otimes\ p^{a+1}} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})\}_{\mathbf{F}_p}$ is *generating*, and, in fact, defines the same splitting over $S^{\mathrm{ord}}_{\mathbf{F}_p}$ as σ_a. That is to say, σ_a extends over $S_{\mathbf{F}_p}$, as desired. This completes the proof of Lemma 4.2. ○

§9. Relation to the Theory of [Mzk1]

So far in this paper, we have only discussed the étale integral structure on Hodge torsors for line bundles of relative degree 1. In [Mzk1], however, we need to make use of Hodge torsors of arbitrary positive degree d. Unfortunately, however, *although the integral structure necessary for what is done in [Mzk1] is implicitly given correctly in [Mzk1] in a neighborhood of infinity — in the form of the "ζ_r^{CG}" of [Mzk1], Chapter V, §4.8, the author is guilty of making a number of erroneous assertions relative to extending this integral structure over the entire moduli stack* $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$, which he would like to take the opportunity to correct in the following discussion:

(1) *First of all, the author wrote [Mzk1], Chapter VI, §1, under the mistaken presumption that the correct integral structure in a neighborhood of infinity (i.e., that given by the "ζ_r^{CG}") coincides with the étale integral structure of the universal extension (i.e., the integral structure presented in §1 of the present paper) in a neighborhood of infinity.* Put another way, the author ignored the fact that the Hodge torsors are only *torsors* over the universal extension, which do not (in general) admit global

integral trivializations (cf. the first Remark following Proposition 3.4). Another more concrete way to describe the author's error is to state that he confused the integral structure defined by the "$\binom{T}{r}$" with that defined by the "$\binom{T-(i_\chi/n)}{r}$" (notation of §4.2).

(2) *The proof that the étale integral structure on the universal extension extends over the entire moduli stack $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ given in [Mzk1], Chapter V, §3, is incomplete.* A complete proof, however, is given in §1 of the present paper.

(3) In fact, in the context of [Mzk1], it is necessary (cf. (1) above) to prove not just that the étale integral structure on the universal extension extends over the entire moduli stack $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$ (i.e., Theorem 1.3 of the present paper), but that *also that the étale integral structure on the Hodge torsors extends over the entire moduli stack* $(\overline{\mathcal{M}}_{1,0})_{\mathbf{Z}}$. This is carried out correctly in the proof of Theorem 4.3 given in the present paper.

The author would like to take this opportunity to apologize to readers of [Mzk1] for these errors.

Next, we would like to clear up the confusion resulting from the above errors by *stating, in the notation of the present paper, the correct integral structure for what is done in [Mzk1]* (more precisely, for the main result (i.e., Theorem A of [Mzk1], Introduction, §1) of [Mzk1]). Since the correct integral structure in a neighborhood at infinity is essentially given accurately in [Mzk1](i.e., using the "ζ_r^{CG}'s" — cf., e.g., the reference to [Mzk1], Chapter V, Theorem 4.8, given in [Mzk1], Introduction, Theorem A, (3)), we concentrate here on the case of *smooth elliptic curves* $E \to S$ over a $\mathbf{Z}$-flat noetherian scheme S.

First, let us recall that in [Mzk1], Chapter V, §2, we defined the object $E^{\dagger}_{[d]}$ by "pushing out" the ω_E-torsor $E^{\dagger}$ via the morphism $[d] : \omega_E \to \omega_E$ given by multiplication by (some positive integer) d. The reason that this was necessary was that we wished to deal with *d-torsion points inside $E^{\dagger}$* which only become *integral* if we use the modified integral structure $E^{\dagger}_{[d]}$. In fact, of course, in order to obtain an integral structure suitable for Theorem A of [Mzk1], Introduction, §1, we must also make further modifications of the integral structure of $E^{\dagger}$. For instance, we would like to use the various properties of the étale integral structure, so, in fact, we need an "étale integral structure version" of $E^{\dagger}_{[d]}$. Since (unlike the kernel of $E^{\dagger} \to E$) the kernel of $E^{\dagger}_{\mathrm{et}} \to E$ is difficult to describe explicitly, instead of defining the étale integral structure version of $E^{\dagger}_{[d]}$ by "pushing out via multiplication by d on the kernel of $E^{\dagger}_{\mathrm{et}} \to E$," we take the following indirect approach: We begin by observing that we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_E & \longrightarrow & E^\dagger & \longrightarrow & E & \longrightarrow & 0 \\
 & & \Big\downarrow [d] & & \Big\downarrow & & \Big\downarrow \mathrm{id} & & \\
0 & \longrightarrow & \omega_E & \longrightarrow & E^\dagger_{[d]} & \longrightarrow & E & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{id} & & \Big\downarrow & & \Big\downarrow [d] & & \\
0 & \longrightarrow & \omega_E & \longrightarrow & E^\dagger & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

of morphisms of group objects over S. This diagram shows that $E^\dagger_{[d]}$ may be defined *either* as the result of pushing forward the first exact sequence via $[d] : \omega_E \to \omega_E$, *or* as the result of pulling back the third exact sequence via $[d] : E \to E$. This makes it natural to define

$$E^\dagger_{[d],\mathrm{et}} \stackrel{\mathrm{def}}{=} E^\dagger_{[d]} \times_{E,[d]} E$$

(i.e., the morphism $E \to E$ implicit in the second factor of the fibered product is the morphism $[d] : E \to E$ given by multiplication by d). Note that we have a natural commutative diagram of morphisms over E

$$\begin{array}{ccc}
E^\dagger_{\mathrm{et}} & \longrightarrow & E^\dagger \\
\Big\downarrow & & \Big\downarrow \\
E^\dagger_{[d],\mathrm{et}} & \longrightarrow & E^\dagger_{[d]}
\end{array}$$

where $F^2(\mathcal{O}_{E^\dagger_{[d],\mathrm{et}}}) = F^2(\mathcal{O}_{E^\dagger_{[d]}})$ (i.e., the subsheaves of the structure sheaves of $E^\dagger_{[d],\mathrm{et}}$ and $E^\dagger_{[d]}$ consisting of functions of torsorial degree < 2 coincide). Moreover, if we consider the morphism $E^\dagger_{[d],\mathrm{et}} \to E^\dagger_{[d]}$ in a neighborhood of infinity, we see that, just as the integral structure of $E^\dagger$ (respectively, $E^\dagger_{[d]}$; $E^\dagger_{\mathrm{et}}$) is defined by the "T^r" (respectively, "$(d \cdot T)^r$"; "$\binom{T}{r}$"), the integral structure of $E^\dagger_{[d],\mathrm{et}}$ is defined by the "$\binom{d \cdot T}{r}$" (cf. the notation of §1).

Now let $\eta \in E(S)$ be a *torsion point of order* m. Then we would like to define the "$E^\dagger_{[d]}$ versions" (i.e., "pushed out by $[d] : \omega_E \to \omega_E$ versions") of E^*, E^*_{et}, where $* \stackrel{\mathrm{def}}{=} \mathcal{O}_E(\eta)$. Since E^* (respectively, E^*_{et}) is a *torsor* over $E^\dagger$ (respectively, $E^\dagger_{\mathrm{et}}$), the natural way to define such objects $E^*_{[d]}$, $E^*_{[d],\mathrm{et}}$ is as follows:

Definition 9.1. We define $E^*_{[d]}$ (respectively, $E^*_{[d],\mathrm{et}}$) is to be the result of executing the "change of structure group" $E^\dagger \to E^\dagger_{[d]}$ (respectively, $E^\dagger_{\mathrm{et}} \to E^\dagger_{[d],\mathrm{et}}$) to the $E^\dagger$- (respectively, $E^\dagger_{\mathrm{et}}$-) torsor E^* (respectively, E^*_{et}).

Thus, in a neighborhood of infinity, (if we let $i_\chi, n \stackrel{\text{def}}{=} 2m$ be as in §4.2, then) just as the integral structure of E^* (respectively, E^*_{et}) is defined by the "$(T-(i_\chi/2m))^r$" (respectively, "$\binom{T-(i_\chi/2m)}{r}$"), the integral structure of $E^*_{[d]}$ (respectively, $E^*_{[d],\text{et}}$) is defined by the "$\{d\cdot(T-(i_\chi/2m))\}^r$" (respectively, "$\binom{d\cdot(T-(i_\chi/2m))}{r}$") (cf. the notation of §1; §4.2).

Theorem 9.2. **(Correct Integral Structure for [Mzk1])** *Let*

$$E \to S$$

be a family of **elliptic curves** *over a* $\mathbf{Z}$*-flat noetherian scheme* S*. Let* $d, m \geq 1$ *be integers such that* m *does not divide* d*. Let* $\eta \in E(S)$ *be a* **torsion point** *of order precisely* m*, and let us write* $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$*. Then the correct definition of the notation "*$(f_S)_*(\overline{\mathcal{L}}|_{E^\dagger_{\infty,[d]}})^{<d}\{\infty, \text{et}\}$*" of [Mzk1], Introduction, §1, Theorem A, is given (relative to the notation of the present paper) by:*

$$F^d(f_*(\mathcal{L}_{E^*_{[d],\text{et}}}))$$

where $* \stackrel{\text{def}}{=} \mathcal{O}_E(\eta)$*;* $F^d(-)$ *denotes the subobject of sections of torsorial degree* $< d$*; and* $E^*_{[d],\text{et}}$ *is as defined in Definition 9.1.*

Remark. Although the definition of the integral structure necessary for the purposes of [Mzk1] that we gave in [Mzk1] is partially inaccurate (as explained above), once one defines this integral structure correctly (as in Theorem 9.2 above), the arguments used to prove Theorem A in [Mzk1], Introduction, §1, are all *valid* without change.

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